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Derived categories, tilted algebras, and Drinfel'd doubles ☆

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Abstract

Given a finite dimensional hereditary algebra Λ over a finite field k, in the derived category $D^b(\Lambda)$ we obtain some formulae on Hall numbers associated to triangles. Using them we prove that, for any tilted algebra Γ of Λ, there is an embedding of the twisted Ringel–Hall algebra of Γ into the Drinfel'd double $\mathcal{D}(\Lambda)$ of the twisted Ringel–Hall algebra of Λ. Furthermore, if Γ is also hereditary, we show that this embedding can be extended as an isomorphism between the two corresponding Drinfel'd doubles. These formulae are also used to construct an automorphism of $\mathcal{D}(\Lambda)$ directly associated to Auslander–Reiten translation, and this automorphism follows by B. Sevenhant–M. Van den Bergh's construction in the case of quivers.

Introduction

The theory of derived categories of finite dimensional algebras developed by Happel is an important tool to study representations of algebras, especially representations of quivers [5]. Recent study shows it also reveals connection between Lie theory and representations of algebras. In [6], Lie algebras of Ringel–Hall type are constructed from the so-called root categories which are some quotients of derived categories of hereditary algebras, and hence a realization of all symmetrizable Kac–Moody algebras is obtained. Some natural structures of derived categories associated with hereditary algebras are

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closely related to the corresponding Lie algebras. For example, the translation functor corresponds to Chevalley involution, and the tilting transformation on the root category implies a decomposition of the Lie algebra [7]. In this paper we try to show that derived categories associated with hereditary algebras also induce some constructions related to corresponding Drinfel'd doubles and quantum groups.

The realization of (the positive part of) quantum groups via Ringel–Hall algebras due to C.M. Ringel and J.A. Green implies that the twisted composition algebra of a quiver (without loops) does not depend on its orientation. For representation theory of quivers, the twisted Ringel–Hall algebra, which contains the composition algebra as its subalgebra, is also an important object to study. As pointed out by G. Lusztig and proven by B. Sevenhant–M. Van den Bergh via Fourier transformations [10], the twisted Ringel–Hall algebra is also independent on the orientation of the quiver. Although variance of orientations induces equivalences between some subcategories of representations of the quiver via the so-called reflection functors due to I.N. Bernstein–I.M. Gel'fand–V.A. Ponomarev [4], these equivalences cannot lead to independence of twisted Ringel–Hall algebras on the orientation of the quiver. However, they were used to show that the Drinfel'd double of the twisted Ringel–Hall algebra does not depend on the orientation of the quiver [10].

M. Auslander, I.M. Platzeck, and I. Reiten generalized reflection functors to more general situations [2]. This generalization leads to tilting theory, which has become an important method in representation theory of artin algebras since then. For an arbitrary finite dimensional hereditary algebra Λ over a finite field k, the motive of this paper is to show that the above result due to B. Sevenhant and M. Van den Bergh can be generalized to any tilting modules. To this end, we use derived category theory to establish some formulae on Hall numbers associated to triangles. It follows that, for an arbitrary tilting module T of Λ, there is an embedding of the twisted Ringel–Hall algebra of the corresponding tilted algebra Γ into the Drinfel'd double $\mathcal{D}(\Lambda)$ of the twisted Ringel–Hall algebra of Λ. In fact, the construction of the embedding follows naturally by tilting theory and derived category theory. More precisely, there is an embedding induced by tilting functors from $\operatorname{mod}\Gamma$ into the derived category $D^b(\Lambda)$ [5], and this embedding induces the embedding of the twisted Ringel–Hall algebra of Γ into $\mathcal{D}(\Lambda)$ in a natural way. In particular, when T is a slice module and hence Γ is hereditary, this embedding can be extended as an isomorphism between Drinfel'd doubles $\mathcal{D}(\Lambda)$ and $\mathcal{D}(\Gamma)$.

For a quiver without loops, the relation between reflection functors and the Auslander–Reiten translation on the representation category of the quiver implies that the AR translation induces an isomorphism between the corresponding Drinfel'd doubles due to Sevenhant–Van den Bergh's construction: one may express the AR translation as a composition of a suitable sequence of reflection functors. Here, for any finite dimensional hereditary algebra Λ over a finite field k, associated to the Auslander–Reiten translation we construct an automorphism of $\mathcal{D}(\Lambda)$ directly.

Such isomorphisms and automorphisms as above can be restricted to the Drinfel'd double $\mathcal{D}_c(\Lambda)$ of the composition algebra of Λ, and hence obtain certain isomorphisms and automorphisms of quantum groups via representation theory of hereditary algebras. Our argument can be applied to a more general situation. In fact, in the derived category $D^b(\Lambda)$ of a hereditary algebra Λ over a finite field k, if there exist two subcategory $\mathcal{T}$

and $\mathcal{F}$ satisfying certain conditions (cf. Theorem 1.1 below), then we can construct an associative algebra $\mathcal{H}(\mathcal{T},\mathcal{F})$ whose structure constants are Hall numbers of triangles in the derived category $D^b(\Lambda)$. By an entirely similar argument to that for tilted algebras, there is an embedding of $\mathcal{H}(\mathcal{T},\mathcal{F})$ into the Drinfel'd double $\mathcal{D}(\Lambda)$. If the category $\mathcal{T}\cup\mathcal{F}$ happens to be hereditary (e.g., a module category of a hereditary algebra), then we have the corresponding Drinfel'd double $\mathcal{D}(\mathcal{T},\mathcal{F})$ according to Green–Ringel theory. Again we can enlarge the above embedding as an isomorphisms between Drinfel'd doubles. This may lead to more isomorphisms of quantum groups via derived categories of hereditary algebras.

Through this paper we assume that Λ is a finite dimensional hereditary algebra over a finite field k and we denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules, and by $D^b(\Lambda) := D^b(\operatorname{mod}\Lambda)$ the derived category of bounded complexes of $\operatorname{mod}\Lambda$. The paper is organized as follows. In Section 1 we establish a formula in the derived category $D^b(\Lambda)$ via analysis of orbits. In Section 2 we fix some notation and review basic construction of Drinfel'd doubles associated with hereditary algebras. In Section 3 we construct the embedding and in Section 4 we extend it as an isomorphism between Drinfel'd doubles for the case that the tilted algebra is hereditary. In Section 5 we construct the automorphism via Auslander–Reiten theory.

1. A formula on triangles of $D^b(\Lambda)$

In a category $\mathcal{C}$, we denote by fg the composition of morphisms $f: A\to B$ and $g: B\to C$ in $\mathcal{C}$. We denote by $(a,b)^{\mathrm{t}}$ the transpose of (a,b). For a finite set S, we denote by $|S|$ the cardinality of S.

Note that the derived category of an associative algebra is a triangulated category. Let us recall some definition and basic facts of triangulated categories based on [5]. Let $\mathcal{C}$ be an additive category with an automorphism $\mathbf{T}$, the so-called *translation functor* of $\mathcal{C}$. Consider sextuples of the form $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}\mathbf{T}X$ for objects X,Y,Z in $\mathcal{C}$. *Morphisms* between sextuples are triples of the form (f,g,h) such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \mathbf{T}X \arrow[d, "\mathbf{T}f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \mathbf{T}X' \end{tikzcd} .$$

Let Ω be a set of some sextuples. The triple $(\mathcal{C},\mathbf{T},\Omega)$ is a *triangulated category* if the following axioms (TR1–4) are satisfied and elements of Ω are called *triangles*.

(TR1) Any sextuple isomorphic to a triangle is a triangle. Any morphism $u: X\to Y$ can be embedded into a triangle $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}\mathbf{T}X$. The sextuple $X\xrightarrow{1_X}X\xrightarrow{0}0\xrightarrow{0}\mathbf{T}X$ is a triangle, where 1_X is the identity morphism of X.

(TR2) If $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}\mathbf{T}X$ is a triangle, then $Y\xrightarrow{v}Z\xrightarrow{w}\mathbf{T}X\xrightarrow{-\mathbf{T}u}\mathbf{T}Y$ is a triangle.

(TR3) For any triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathbf{T}X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \mathbf{T}X'$, if there are morphisms $f : X \to X'$, $g : Y \to Y'$ such that $fu' = ug$, then there is a morphism (f, g, h) from the first triangle to the second.

(TR4) (The octahedral axiom). Assume that

$$X \xrightarrow{u} Y \xrightarrow{i} Z \xrightarrow{i'} \mathbf{T}X, \qquad Y \xrightarrow{v} Z \xrightarrow{j} X' \xrightarrow{j'} \mathbf{T}Y, \qquad X \xrightarrow{uv} Z \xrightarrow{k} Y' \xrightarrow{k'} \mathbf{T}X$$

are triangles. Then there are morphisms $f : Z' \to Y'$ and $g : Y' \to X'$ such that the following diagram commutes and the third row is a triangle:

$$\begin{tikzcd}
\mathbf{T}^{-1}Y' \arrow[r, "\mathbf{T}^{-1}k'"] \arrow[d, "\mathbf{T}^{-1}g"] & X \arrow[r, equal] \arrow[d, "u"] & X \arrow[d, "uv"] & & \\
\mathbf{T}^{-1}X' \arrow[r, "\mathbf{T}^{-1}j'"] & Y \arrow[r, "v"] \arrow[d, "i"] & Z \arrow[r, "j"] \arrow[d, "k"] & X' \arrow[r, "j'"] \arrow[d, equal] & \mathbf{T}Y \arrow[d, "\mathbf{T}i"] \\
& Z' \arrow[r, "f"] \arrow[d, "i'"] & Y' \arrow[r, "g"] \arrow[d, "k'"] & X' \arrow[r, "j'(\mathbf{T}i)"] & \mathbf{T}Z' \\
& \mathbf{T}X \arrow[r, equal] & \mathbf{T}X & &
\end{tikzcd}.$$

For simplicity a triangulated category $(\mathcal{C}, \mathbf{T}, \Omega)$ will be denoted as $\mathcal{C}$. The following basic facts of triangulated categories (cf. [5]) will be used in sequel.

Proposition 1.1. *For a triangulated category $\mathcal{C}$, we have that*

(1) *If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathbf{T}X$ is a triangle, then $uv = vw = 0$.*
(2) *If (f, g, h) is a morphism between triangles such that f and g is an isomorphism, then h is also an isomorphism.*
(3) *For any $M \in \mathcal{C}$, the functors $\mathbf{H} = \mathrm{Hom}(M, -)$ and $\mathbf{G} = \mathrm{Hom}(-, M)$ give rise to long exact sequences for any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathbf{T}X$:*

$$\cdots \longrightarrow \mathbf{H}(\mathbf{T}^i X) \xrightarrow{\mathbf{H}(\mathbf{T}^i u)} \mathbf{H}(\mathbf{T}^i Y) \xrightarrow{\mathbf{H}(\mathbf{T}^i v)} \mathbf{H}(\mathbf{T}^i Z) \xrightarrow{\mathbf{H}(\mathbf{T}^i w)} \mathbf{H}(\mathbf{T}^{i+1} X) \longrightarrow \cdots$$

and

$$\cdots \longleftarrow \mathbf{G}(\mathbf{T}^i X) \xleftarrow{\mathbf{G}(\mathbf{T}^i u)} \mathbf{G}(\mathbf{T}^i Y) \xleftarrow{\mathbf{G}(\mathbf{T}^i v)} \mathbf{G}(\mathbf{T}^i Z) \xleftarrow{\mathbf{G}(\mathbf{T}^i w)} \mathbf{G}(\mathbf{T}^{i+1} X) \longleftarrow \cdots.$$

(4) *If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathbf{T}X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \mathbf{T}X'$ are triangles and $g : Y \to Y'$ is a morphism such that $ugv' = 0$, then there exist some morphism (f, g, h) from the first triangle to the second, and if $\mathrm{Hom}(X, \mathbf{T}^{-1}Z') = 0$, then f and h are uniquely determined by g.*

(5) *Assume that* $\mathcal{C} = D^b(A)$, *the derived category of an associative algebra* A. *Then for any* A-*modules* X, Y *we have* $\mathrm{Hom}_{D^b(A)}(\mathbf{T}^i X, Y) = 0$ *and* $\mathrm{Hom}_{D^b(A)}(X, \mathbf{T}^i Y) \simeq \mathrm{Ext}^i_A(X, Y)$ *for* $i > 0$.

We have the following

Proposition 1.2. *Let* $\mathcal{C}$ *be a finite triangulated category with translation functor* $\mathbf{T}$, *i.e.,* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *is a finite set for any* $X, Y \in \mathcal{C}$. *Given triangles* $L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w} \mathbf{T}L$ *and* $L' \xrightarrow{u'} M' \xrightarrow{v'} N' \xrightarrow{w'} \mathbf{T}L'$ *such that*

$$\mathrm{Hom}_{\mathcal{C}}(M, L') = \mathrm{Hom}_{\mathcal{C}}(N, L') = \mathrm{Hom}_{\mathcal{C}}(\mathbf{T}M, M') = 0$$

and there is a morphism (f, g, h) *from the first triangle to the second, then, for the fixed* f, g, *the number of possible* h*'s is* $|\mathrm{Hom}_{\mathcal{C}}(\mathbf{T}L, M')|$.

Proof. It suffices to count the number of morphisms $h_0 : N \to N'$ such that $vh_0 = 0$ and $h_0 w' = 0$. Let H be the set of such h_0's. Define a map

$$\rho : \mathrm{Hom}_{\mathcal{C}}(\mathbf{T}L, M') \to \mathrm{H} : d \mapsto wdv'.$$

Note that for any $d : \mathbf{T}L \to M'$, we indeed have that wdv' belongs to H since $vwdv' = wdv'w' = 0$ by (1) of Proposition 1.1. Assume that $wdv' = 0$. Then, by (4) of Proposition 1.1 we have that $wd = 0$ since $\mathrm{Hom}_{\mathcal{C}}(N, L') = 0$. By (3) of Proposition 1.1 and $\mathrm{Hom}_{\mathcal{C}}(\mathbf{T}M, M') = 0$ there is an exact sequence $0 \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{T}L, M') \to \mathrm{Hom}_{\mathcal{C}}(N, M')$ which means that $d = 0$. So, ρ is injective. For any $h_0 \in \mathrm{H}$, by (3) of Proposition 1.1 and $\mathrm{Hom}_{\mathcal{C}}(N, L') = 0$, we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(N, M') \to \mathrm{Hom}_{\mathcal{C}}(N, N') \to \mathrm{Hom}_{\mathcal{C}}(N, \mathbf{T}L').$$

So, there is a unique $\delta : N \to M'$ such that $h_0 = \delta v'$ since $h_0 w = 0$. Thus, $v\delta v' = vh_0 = 0$. By a similar argument as above we have that $v\delta = 0$ since $\mathrm{Hom}_{\mathcal{C}}(M, L') = 0$, and we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{T}L, M') \to \mathrm{Hom}_{\mathcal{C}}(N, M') \to \mathrm{Hom}_{\mathcal{C}}(M, M'),$$

since $\mathrm{Hom}_{\mathcal{C}}(\mathbf{T}M, M') = 0$. It follows that there is a unique $d : \mathbf{T}L \to M'$ such that $\delta = wd$ and hence $h_0 = wdv'$, which means that ρ is surjective. This completes the proof. $\square$

Now we prove the following

Lemma 1.1. *Let* $\mathcal{C}$ *be a triangulated category with translation functor* $\mathbf{T}$. *If there is a triangle* $X \xrightarrow{(f, f')} A \oplus B \xrightarrow{(g, g)^{\mathrm{t}}} Y \xrightarrow{h} \mathbf{T}X$, *then there exists a unique object* C *up to isomorphism in* $\mathcal{C}$ *such that there is a morphism between triangles*

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "f'"] & A \arrow[r, "v"] \arrow[d, "-g"] & \mathbf{T}C \arrow[d, equal] \\ C \arrow[r, "u'"] & B \arrow[r, "g'"] & Y \arrow[r, "v'"] & \mathbf{T}C \end{tikzcd} \tag{$*$}$$

for some morphisms u, u', v, v' with $h = v'(\mathbf{T}u)$. On the other hand, if there is a morphism between triangles as above, then there is a triangle of the form

$$X \xrightarrow{(f,f')} A \oplus B \xrightarrow{(\bar{g},g')^{\mathrm{t}}} Y \xrightarrow{v'(\mathbf{T}u)} \mathbf{T}X$$

for some morphism $\bar{g}$.

Proof. By (TR1) and (2) of Proposition 1.1, there is a unique object C up to isomorphism in $\mathcal{C}$ such that there is a triangle $C \xrightarrow{u} X \xrightarrow{f} A \xrightarrow{v} \mathbf{T}C$ for some morphisms u, v. Hence, $C \xrightarrow{-u} X \xrightarrow{-f} A \xrightarrow{v} \mathbf{T}C$ is also a triangle by (TR1). Since we have a splitting triangle $A \oplus B \xrightarrow{(-1,0)^{\mathrm{t}}} A \xrightarrow{0} \mathbf{T}B \xrightarrow{(0,1)} \mathbf{T}A \oplus \mathbf{T}B$, by (TR4) there exist morphisms v' and u' such that the following diagram commutes and the third row is a triangle:

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "u'"'] & X \arrow[r, equal] \arrow[d, "{(f,f')}"] & X \arrow[d, "-f"] & & \\ B \arrow[r, "{(0,1)}"] & A \oplus B \arrow[r, "{(-1,0)^{\mathrm{t}}}"] \arrow[d, "{(g,g')^{\mathrm{t}}}"] & A \arrow[r, "0"] \arrow[d, "v"] & \mathbf{T}B \arrow[r, "{(0,1)}"] \arrow[d, equal] & \mathbf{T}A \oplus \mathbf{T}B \arrow[d, "{(\mathbf{T}g,\mathbf{T}g')^{\mathrm{t}}}"] \\ & Y \arrow[r, "v'"] \arrow[d, "h"] & \mathbf{T}C \arrow[r, "\mathbf{T}u'"] \arrow[d, "\mathbf{T}u"] & \mathbf{T}B \arrow[r, "\mathbf{T}g'"] & \mathbf{T}Y \\ & \mathbf{T}X \arrow[r, equal] & \mathbf{T}X & & \end{tikzcd} .$$

So, by (TR2) we have a triangle $C \xrightarrow{-u'} B \xrightarrow{-g'} Y \xrightarrow{v'} \mathbf{T}C$, and by (TR1), we have a triangle $C \xrightarrow{u'} B \xrightarrow{g'} Y \xrightarrow{v'} \mathbf{T}C$. Since $gv' = v$, $uf' = u'$, $h = v'(\mathbf{T}u)$, and $f(-g) = f'g'$ (by (1) of Proposition 1.1), we obtain the required morphism $(*)$ between triangles.

Now assume that there is a morphism as in $(*)$. Then, by (TR1) we have a triangle $X \xrightarrow{(f,f')} A \oplus B \xrightarrow{(g_1,g_1')^{\mathrm{t}}} \overline{Y} \xrightarrow{\bar{h}} \mathbf{T}X$ for some objects $\overline{Y}$ and morphisms g_1, g_1'. By the proof of former part we have a morphism between triangles:

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "f'"] & A \arrow[r, "v"] \arrow[d, "-g_1"] & \mathbf{T}C \arrow[d, equal] \\ C \arrow[r, "u_1'"] & B \arrow[r, "g_1'"] & \overline{Y} \arrow[r, "v_1'"] & \mathbf{T}C \end{tikzcd} ,$$

where $\bar{h} = v_1'(\mathbf{T}u)$. Note that, by $(*)$, $u_1' = uf' = u'$. So, by (TR3) there is a morphism $(1_C, 1_B, y)$ from the triangle $C \xrightarrow{u_1'} B \xrightarrow{g_1'} \overline{Y} \xrightarrow{v_1'} \mathbf{T}C$ to the triangle $C \xrightarrow{u'} B \xrightarrow{g'} Y \xrightarrow{v'} \mathbf{T}C$ for some morphism $y : \overline{Y} \to Y$. Thus, $g' = g_1'y$. By (2) of Proposition 1.1, y is an isomorphism. Set $\bar{g} = g_1 y$. Hence there is an isomorphism $(1_X, 1_{A\oplus B}, y)$ from $X \xrightarrow{(f,f')} A \oplus B \xrightarrow{(g_1,g_1')^{\mathrm{t}}} \overline{Y} \xrightarrow{\bar{h}} \mathbf{T}X$ to the sextuple $X \xrightarrow{(f,f')} A \oplus B \xrightarrow{(\bar{g},g')^{\mathrm{t}}} Y \xrightarrow{h'} \mathbf{T}X$, where $h' = y^{-1}\bar{h} = y^{-1}v_1'(\mathbf{T}u) = y^{-1}yv'(\mathbf{T}u) = v'(\mathbf{T}u)$. By (TR1), the latter is a triangle as required. This completes the proof. □

For a finite dimensional hereditary algebra Λ over a finite field k, let $D^b(\Lambda)$ be the derived category of *bounded* complex category of $\operatorname{mod}\Lambda$ and denote the translation functor of $D^b(\Lambda)$ by $\mathbf{T}$. Fix a full embedding of $\operatorname{mod}\Lambda$ into $D^b(\Lambda)$. For $X, Y, Z \in D^b(\Lambda)$, set

$$W(X, Y; Z) = \bigl\{(f, g, h):\ X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{h} \mathbf{T}X \text{ is a triangle}\bigr\}.$$

Consider the action of $\operatorname{Aut} X \times \operatorname{Aut} Y$ on $W(X, Y; Z)$ given by

$$(\alpha, \beta)(f, g, h) = \bigl(\alpha^{-1} f, g\beta, \beta^{-1} h(\mathbf{T}\alpha)\bigr) \tag{1.1}$$

for $(\alpha, \beta) \in \operatorname{Aut} X \times \operatorname{Aut} Y$ and $(f, g, h) \in W(X, Y; Z)$. Denote the orbit space by $V(X, Y; Z)$. Note that the cardinality $|V(X, Y; Z)|$ of $V(X, Y; Z)$ is just the analogue of *Hall numbers* (see Section 2 below) in the sense of Ringel (cf. [6,8]). In fact, since Λ is hereditary, for any objects X, Y, Z in $\operatorname{mod}\Lambda$, we have that

$$\bigl|W(X, Y; Z)\bigr| = \bigl|V(X, Y; Z)\bigr| |\operatorname{Aut} X| |\operatorname{Aut} Y|$$

and $|V(X, Y; Z)|$ equals to the Hall number $\mathbf{F}^Z_{YX}$. The rest of this section is devoted to deducing a formula on such numbers (see Theorem 1.1 below). We also need the following technical lemmas.

Lemma 1.2. *For objects $A, B, X, Y \in \operatorname{mod}\Lambda$, if there are triangles in $D^b(\Lambda)$ of the form $C \to X \to A \to \mathbf{T}C$ and $B \to Y \to C \to \mathbf{T}C$, then C belongs to* $\operatorname{mod}\Lambda$.

Proof. Since Λ is hereditary, by a similar argument as in the proof of Corollary 5.3 of Chapter I in [5], we may assume that $C = \bigoplus_{i\in\mathbf{Z}} \mathbf{T}^i C_i$ for some $C_i \in \operatorname{mod}\Lambda$. Then the result follows by Lemma 2 of Section 2 in [6]. □

For $X, Y, A, B, C \in D^b(\Lambda)$, consider the action of $\operatorname{Aut}\mathbf{T}Y$ on $W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)$ given by

$$y\bigl((f, f'), (g, g')^{\mathrm{t}}, h\bigr) = \bigl((f, f'), \bigl(g_1, g_1'\bigr)^{\mathrm{t}}, h_1\bigr) \tag{1.2}$$

for $y \in \operatorname{Aut} Y$ and $((f, f'), (g, g')^{\mathrm{t}}, h) \in W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)$, where g_1, g_1', and h_1 fit into the following commutative diagram with rows being triangles:

$$\begin{tikzcd}
X \arrow[r, "{(f,f')}"] \arrow[d, equal] & A \oplus \mathbf{T}B \arrow[r, "{(g,g')^{\mathrm{t}}}"] \arrow[d, equal] & \mathbf{T}Y \arrow[r, "h"] \arrow[d, "\mathbf{T}y"] & \mathbf{T}X \arrow[d, equal] \\
X \arrow[r, "{(f,f')}"] & A \oplus \mathbf{T}B \arrow[r, "{(g_1,g_1')^{\mathrm{t}}}"] & \mathbf{T}Y \arrow[r, "h_1"] & \mathbf{T}X
\end{tikzcd}.$$

Denote the corresponding orbit space by $W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)^{\vee}$. Also consider the action of $\operatorname{Aut} C \times \operatorname{Aut} \mathbf{T}Y$ on $W(C, A; X) \times W(C, \mathbf{T}Y; \mathbf{T}B)$ given by

$$(c, y)\big((u, f, v), (u', g', v')\big) = \big((u_1, f, v_1), \big(u_1', g_1', v_1'\big)\big) \tag{1.3}$$

for any $(c, y) \in \operatorname{Aut} C \times \operatorname{Aut} Y$ and $((u, f, v), (u', g', v')) \in W(C, A; X) \times W(C, \mathbf{T}Y; \mathbf{T}B)$, where morphisms $u_1, v_1, u_1', g_1', v_1'$ fit into the following two commutative diagrams with rows being triangles:

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "c"'] & X \arrow[r, "f"] \arrow[d, equal] & A \arrow[r, "v"] \arrow[d, equal] & \mathbf{T}C \arrow[d, "\mathbf{T}c"] \\
C \arrow[r, "u_1"] & X \arrow[r, "f"] & A \arrow[r, "v_1"] & \mathbf{T}C
\end{tikzcd}, \qquad
\begin{tikzcd}
C \arrow[r, "u'"] \arrow[d, "c"'] & \mathbf{T}B \arrow[r, "g'"] \arrow[d, equal] & \mathbf{T}Y \arrow[r, "v'"] \arrow[d, "\mathbf{T}y"] & \mathbf{T}C \arrow[d, "\mathbf{T}c"] \\
C \arrow[r, "u_1'"] & \mathbf{T}B \arrow[r, "g_1'"] & \mathbf{T}Y \arrow[r, "v_1'"] & \mathbf{T}C
\end{tikzcd}.$$

Denote the corresponding orbit space by $(W(C, A; X) \times W(C, \mathbf{T}Y; \mathbf{T}B))^{\vee}$.

Keep notation as above. We have the following

Lemma 1.3. *For $X, Y \in D^b(\Lambda)$ and $A, B \in \operatorname{mod} \Lambda$, there is a surjective map*

$$\Phi : W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)^{\vee} \to \bigcup \big(W(C, A; X) \times W(C, \mathbf{T}Y; \mathbf{T}B)\big)^{\vee},$$

where the union is taken over the isoclasses of C such that there are triangles of the form $C \to X \to A \to \mathbf{T}C$ and $B \to Y \to C \to \mathbf{T}B$, and Φ is given by

$$\big((f, f'), (g, g')^{\mathrm{t}}, h\big)^{\vee} \mapsto \big((u, f, v), (u', g', v')\big)^{\vee}$$

such that there is a morphism between triangles as follows:

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "f'"] & A \arrow[r, "v"] \arrow[d, "-g"] & \mathbf{T}C \arrow[d, equal] \\
C \arrow[r, "u'"] & \mathbf{T}B \arrow[r, "g'"] & \mathbf{T}Y \arrow[r, "v'"] & \mathbf{T}C
\end{tikzcd}. \tag{$**$}$$

Moreover, if $\mathrm{Hom}_{D^b(\Lambda)}(X,B)=0$*, then for any*

$$\big((u,f,v),(u',g',v')\big)^\vee \in \big(W(C,A;X)\times W(C,\mathbf{T}Y;\mathbf{T}B)\big)^\vee,$$

we have that

$$\big|\Phi^{-1}\big(\big((u,f,v),(u',g',v')\big)^\vee\big)\big| = \big|\mathrm{Hom}_{D^b(\Lambda)}(A,\mathbf{T}B)\big|\big|\mathrm{Hom}_{D^b(\Lambda)}(\mathbf{T}C,\mathbf{T}B)\big|^{-1}.$$

Proof. By Lemma 1.1, for $((f,f'),(g,g')^{\mathrm{t}},h)^\vee \in W(X,\mathbf{T}Y;A\oplus\mathbf{T}B)^\vee$, such a morphism $(**)$ between triangles exists. We claim that Φ is well-defined. Assume that there is a $y_1\in \mathrm{Aut}\,Y$ such that, under the action of y_1 given by (1.2),

$$\big((f_1,f_1'),(g_1,g_1')^{\mathrm{t}},h_1\big) = y_1\big((f,f'),(g,g')^{\mathrm{t}},h\big).$$

Since $f=f_1$ and y_1 is an isomorphism, by (TR3) and (2) of Proposition 1.1, there exist some $c_1,c_2\in\mathrm{Aut}\,C$ such that the following two diagrams commute:

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "c_1"'] & X \arrow[r, "f"] \arrow[d, equal] & A \arrow[r, "v"] \arrow[d, equal] & \mathbf{T}C \arrow[d, "\mathbf{T}c_1"] \\ C \arrow[r, "u_1"] & X \arrow[r, "f_1"] & A \arrow[r, "v_1"] & \mathbf{T}C \end{tikzcd},\qquad \begin{tikzcd} C \arrow[r, "u'"] \arrow[d, "c_2"'] & \mathbf{T}B \arrow[r, "g'"] \arrow[d, equal] & \mathbf{T}Y \arrow[r, "v'"] \arrow[d, "\mathbf{T}y"] & \mathbf{T}C \arrow[d, "\mathbf{T}c_2"] \\ C \arrow[r, "u_1'"] & \mathbf{T}B \arrow[r, "g_1'"] & \mathbf{T}Y \arrow[r, "v_1'"] & \mathbf{T}C \end{tikzcd}.$$

Here the morphisms u,v,u',v' and u_1,v_1,u_1',v_1' are given by Lemma 1.1. In view of (1.3), it remains to show that there exist $c\in\mathrm{Aut}\,C$ and $y\in\mathrm{Aut}\,Y$ such that

$$\big((u_1,f,v_1),(u_1',g',v_1')\big) = \big(\big(c^{-1}u,f,v(\mathbf{T}c)\big),\big(c^{-1}u',g'(\mathbf{T}y),(\mathbf{T}y)^{-1}v'(\mathbf{T}c)\big)\big).$$

Set $c=c_1$. Then $u_1'=u_1f_1'=u_1f'=c_1^{-1}uf'=c_1^{-1}u'=c^{-1}u'$. On the other hand, by (1.3) again, we have that $u_1'=c^{-1}c_2u_1'$. Thus, by (TR3) and (2) of Proposition 1.1 there exist some $y_2\in\mathrm{Aut}\,Y$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "u_1'"] \arrow[d, "c_2^{-1}c"'] & \mathbf{T}B \arrow[r, "g_1'"] \arrow[d, equal] & \mathbf{T}Y \arrow[r, "v_1'"] \arrow[d, "\mathbf{T}y_2"] & \mathbf{T}C \arrow[d, "\mathbf{T}c_2^{-1}\mathbf{T}c"] \\ C \arrow[r, "u_1'"] & \mathbf{T}B \arrow[r, "g_1'"] & \mathbf{T}Y \arrow[r, "v_1'"] & \mathbf{T}C \end{tikzcd}.$$

So we have that $g'(\mathbf{T}y_1\mathbf{T}y_2)=g_1'(\mathbf{T}y_2)=g_1'$ and

$$(\mathbf{T}y_1\mathbf{T}y_2)v_1' = (\mathbf{T}y_1)v_1'(\mathbf{T}c_2)^{-1}\mathbf{T}c = v'\mathbf{T}c_2\mathbf{T}c_2^{-1}\mathbf{T}c = v'\mathbf{T}c.$$

So $y=y_1y_2$ is as required. Thus we have proven that

$$\Phi\big(\big((f,f'),(g,g')^{\mathrm{t}},h\big)^\vee\big) = \Phi\big(\big((f_1,f_1'),(g_1,g_1')^{\mathrm{t}},h_1\big)^\vee\big),$$

which means that Φ is well-defined. For

$$\big((u, f, v), (u', g', v')\big)^{\vee} \in \big(W(C, A; X) \times W(C, \mathbf{T}Y; \mathbf{T}B)\big)^{\vee},$$

by (5) of Proposition 1.1 we have

$$\mathrm{Hom}_{D^b(\Lambda)}\big(\mathbf{T}^{-1}A, \mathbf{T}B\big) \simeq \mathrm{Ext}^2_{\Lambda}(A, B) = 0$$

since Λ is hereditary, and by (4) of Proposition 1.1, there exist f' and g such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{T}^{-1}A \arrow[r] & C \arrow[r, "u"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "f'"] & A \arrow[r, "v"] \arrow[d, "-g"] & \mathbf{T}C \arrow[d, equal] \\
 & C \arrow[r, "u'"] & \mathbf{T}B \arrow[r, "g'"] & \mathbf{T}Y \arrow[r, "v'"] & \mathbf{T}C
\end{tikzcd} .$$

By Lemma 1.1, we have a triangle

$$X \xrightarrow{(f, f')^{\mathrm{t}}} A \oplus \mathbf{T}B \xrightarrow{(\bar{g}, g')} \mathbf{T}Y \xrightarrow{h} \mathbf{T}X,$$

where $h = v'(\mathbf{T}u)$. Set

$$\Phi\Big(\big((f, f')^{\mathrm{t}}, (g, g'), h\big)^{\vee}\Big) = \big((u_1, f, v_1), (u_1', g', v_1')\big)^{\vee}.$$

Thus there is a morphism between triangles as follows:

$$\begin{tikzcd}
C \arrow[r, "u_1"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "f'"] & A \arrow[r, "v_1"] \arrow[d, "-g'"] & \mathbf{T}C \arrow[d, equal] \\
C \arrow[r, "u_1'"] & \mathbf{T}B \arrow[r, "g'"] & \mathbf{T}Y \arrow[r, "v_1'"] & \mathbf{T}C
\end{tikzcd} .$$

Comparing with $(**)$, by (TR3) and (2) of Proposition 1.1, there are $c_1, c_2 \in \mathrm{Aut}\, C$ such that the following diagrams commutes:

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "c_1"'] & X \arrow[r, "f"] \arrow[d, equal] & A \arrow[r, "v"] \arrow[d, equal] & \mathbf{T}C \arrow[d, "\mathbf{T}c_1"] \\
C \arrow[r, "u_1"] & X \arrow[r, "f"] & A \arrow[r, "v_1"] & \mathbf{T}C
\end{tikzcd} , \qquad
\begin{tikzcd}
C \arrow[r, "u'"] \arrow[d, "c_2"'] & \mathbf{T}B \arrow[r, "g'"] \arrow[d, equal] & \mathbf{T}Y \arrow[r, "v'"] \arrow[d, equal] & \mathbf{T}C \arrow[d, "\mathbf{T}c_2"] \\
C \arrow[r, "u_1'"] & \mathbf{T}B \arrow[r, "g'"] & \mathbf{T}Y \arrow[r, "v_1'"] & \mathbf{T}C
\end{tikzcd} .$$

Since $u_1' = u_1 f' = c_1^{-1} u f' = c_1^{-1} u' = c_1^{-1} c_2 u_1'$, by (TR3) and (2) of Proposition 1.1 there exist some $y \in \operatorname{Aut} Y$ such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "u_1'"] \arrow[d, "c_2^{-1}c_1"'] & \mathbf{T}B \arrow[r, "g'"] \arrow[d, equal] & \mathbf{T}Y \arrow[r, "v_1'"] \arrow[d, "\mathbf{T}y"] & \mathbf{T}C \arrow[d, "\mathbf{T}c_2^{-1}\mathbf{T}c_1"] \\
C \arrow[r, "u_1'"] & \mathbf{T}B \arrow[r, "g'"] & \mathbf{T}Y \arrow[r, "v_1'"] & \mathbf{T}C
\end{tikzcd}.$$

Since $(\mathbf{T}y)v_1' = v_1' \mathbf{T}c_2' \mathbf{T}c_1 = v' \mathbf{T}c_1$, by (1.3), we have that

$$(c_1, y)\big((u, f, v), (u', g', v')\big) = \big((u_1, f, v_1), (u_1', g', v_1')\big),$$

which means that $((u, f, v), (u', g', v'))^\vee = ((u_1, f, v_1), (u_1', g', v_1'))^\vee$ and hence we have proven that Φ is surjective.

For any

$$\big((u, f, v), (u', g', v')\big)^\vee \in \big(W(C, A; X) \times W(C, \mathbf{T}Y; \mathbf{T}B)\big)^\vee,$$

by definition of Φ the fiber $\Phi^{-1}(((u, f, v), (u', g', v'))^\vee)$ is

$$\Big\{\big((f, f'), (g, g')^{\mathrm{t}}, v'(\mathbf{T}u)\big)^\vee \;\Big|\; X \xrightarrow{(f,f')} A \oplus \mathbf{T}B \xrightarrow{(g,g')^{\mathrm{t}}} \mathbf{T}Y \xrightarrow{v'(\mathbf{T}u)} \mathbf{T}X \text{ is a triangle}\Big\}.$$

Let S be the set of morphisms $f' : X \to \mathbf{T}B$ such that $f'u = u'$. By (TR3), for any $f' \in S$ there exist some morphisms $(1_C, f', -g)$ from $C \xrightarrow{u} X \xrightarrow{f} A \xrightarrow{v} \mathbf{T}C$ to $C \xrightarrow{u'} \mathbf{T}B \xrightarrow{g'} \mathbf{T}Y \xrightarrow{v'} \mathbf{T}C$ and hence some $((f, f'), (g, g')^{\mathrm{t}}, v'(\mathbf{T}u))^\vee$, which belong to the fiber. Note that, by the definition of the action of $\operatorname{Aut}\mathbf{T}Y$ given by (1.2), for the fixed f, g', different f' determines different elements in $\Phi^{-1}(((u, f, v), (u', g', v'))^\vee)$. Thus we have that

$$\Phi^{-1}\Big(\big((u, f, v), (u', g', v')\big)^\vee\Big) = |S|.$$

By (4) of Proposition 1.1 and applying $\operatorname{Hom}_{D^b(\Lambda)}(-, \mathbf{T}B)$ to the triangle $C \xrightarrow{u} X \xrightarrow{f} A \xrightarrow{v} \mathbf{T}C$ we have the following exact sequence:

$$\begin{aligned}
0 &\to \operatorname{Hom}_{D^b(\Lambda)}(\mathbf{T}C, \mathbf{T}B) \to \operatorname{Hom}_{D^b(\Lambda)}(A, \mathbf{T}B) \\
&\to \operatorname{Hom}_{D^b(\Lambda)}(X, \mathbf{T}B) \xrightarrow{\operatorname{Hom}(u,\mathbf{T}B)} \operatorname{Hom}_{D^b(\Lambda)}(C, \mathbf{T}B) \to 0
\end{aligned}$$

since $\operatorname{Hom}_{D^b(\Lambda)}(X, B) = 0$. So we have that

$$|S| = \big|\operatorname{Ker}\operatorname{Hom}_{D^b(\Lambda)}(u, \mathbf{T}B)\big| = \big|\operatorname{Hom}_{D^b(\Lambda)}(A, \mathbf{T}B)\big|\big|\operatorname{Hom}_{D^b(\Lambda)}(\mathbf{T}C, \mathbf{T}B)\big|^{-1},$$

which completes the proof. $\square$

Theorem 1.1. *Assume that there are full subcategories $\mathcal{T}$ and $\mathcal{F}$ of* $\operatorname{mod}\Lambda$ *such that there is no nonzero Λ-morphism from $\mathcal{T}$ to $\mathcal{F}$. Then, for $X, A \in \mathcal{T}$ and $Y, B \in \mathcal{F}$, we have that*

$$\begin{aligned}&\big|V(X, \mathbf{T}Y; A \oplus \mathbf{T}B)\big||\operatorname{Aut} X||\operatorname{Aut} Y|\\ &\quad = \sum \big|V(C, A; X)\big|\big|V(B, C; Y)\big||\operatorname{Aut} C||\operatorname{Aut} B||\operatorname{Aut} A|\big|\operatorname{Hom}_{D^b(\Lambda)}(A, \mathbf{T}B)\big|, \qquad (1.4)\end{aligned}$$

where the sum is taken over the isoclasses of C such that there are triangles of the form $C \to X \to A \to \mathbf{T}C$ and $B \to Y \to C \to \mathbf{T}B$. Moreover, such C's belong to $\operatorname{mod}\Lambda$.

Proof. By Lemma 1.2 such objects C belong to $\operatorname{mod}\Lambda$. Thus, by (4) of Proposition 1.1, for any fixed $y \in \operatorname{Aut} Y$, the action of $\operatorname{Aut} C \times \{\mathbf{T}y\}$ on $W(C, A; X) \times W(C, \mathbf{T}Y; \mathbf{T}B)$ given by (1.3) is free, since $\operatorname{Hom}_{D^b(\Lambda)}(\mathbf{T}B, C) = 0$ by (5) of Proposition 1.1. On the other hand, for fixed morphisms u', g', v', by Proposition 1.2 the number of possible isomorphisms $y \in \operatorname{Aut} Y$ to make the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "u'"] \arrow[d, equal] & \mathbf{T}B \arrow[r, "g'"] \arrow[d, equal] & \mathbf{T}Y \arrow[r, "v'"] \arrow[d, "\mathbf{T}y"] & \mathbf{T}C \arrow[d, equal] \\ C \arrow[r, "u'"] & \mathbf{T}B \arrow[r, "g'"] & \mathbf{T}Y \arrow[r, "v'"] & \mathbf{T}C \end{tikzcd}$$

is $|\operatorname{Hom}_{D^b(\Lambda)}(\mathbf{T}C, \mathbf{T}B)|$. So, by the action of $\operatorname{Aut} C \times \operatorname{Aut} \mathbf{T}Y$, we have that

$$\begin{aligned}&\big|\big(W(C, A; X) \times W(C, \mathbf{T}Y; \mathbf{T}B)\big)^\vee\big|\\ &\quad = \big|W(C, A; X)\big|\big|W(C, \mathbf{T}Y; \mathbf{T}B)\big|\big|\operatorname{Hom}_{D^b(\Lambda)}(\mathbf{T}C, \mathbf{T}B)\big||\operatorname{Aut} C|^{-1}|\operatorname{Aut} \mathbf{T}Y|^{-1}.\end{aligned}$$

Since $\operatorname{Hom}_{D^b(\Lambda)}(X, B) = 0$, by Lemma 1.3, we have that

$$\begin{aligned}&\big|W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)^\vee\big|\\ &\quad = \big|\operatorname{Hom}_{D^b(\Lambda)}(A, \mathbf{T}B)\big| \sum \big|W(C, A; X)\big|\big|W(C, \mathbf{T}Y; \mathbf{T}B)\big||\operatorname{Aut} C|^{-1}|\operatorname{Aut} \mathbf{T}Y|^{-1},\end{aligned}$$

where the sum is taken over isoclasses of C as in Lemma 1.3. Since $\operatorname{Hom}_{D^b(\Lambda)}(\mathbf{T}X, \mathbf{T}Y) \simeq \operatorname{Hom}_{D^b(\Lambda)}(X, Y) = 0$, by (4) of Proposition 1.1, if there is a morphism $(1_X, 1_{A\oplus\mathbf{T}B}, \mathbf{T}y)$ from the triangle

$$X \xrightarrow{(f, f')} A \oplus \mathbf{T}B \xrightarrow{(g, g')^{\mathrm{t}}} \mathbf{T}Y \xrightarrow{h} \mathbf{T}X$$

to itself, then y must be the identity morphism, which means the action of $\operatorname{Aut} \mathbf{T}Y$ on $W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)$ (1.2) is free. So we have that

$$\big|W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)^\vee\big| = \big|W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)\big||\operatorname{Aut} \mathbf{T}Y|^{-1}.$$

By the same reason, we see that the action of $\mathrm{Aut}\, X \times \mathrm{Aut}\, \mathbf{T}Y$ on $W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)$ via (1.1) is free. So we have that

$$\left|W(X, \mathbf{T}Y; A \oplus \mathbf{T}B)\right| = |\mathrm{Aut}\, X||\mathrm{Aut}\, \mathbf{T}Y|\left|V(X, \mathbf{T}Y; A \oplus \mathbf{T}B)\right|.$$

Since A, B, C, X, Y belong to $\mathrm{mod}\, \Lambda$, we also have that

$$\left|W(C, A; X)\right| = |\mathrm{Aut}\, A||\mathrm{Aut}\, C|\left|V(C, A; X)\right|,$$

and

$$\left|W(C, \mathbf{T}Y; \mathbf{T}B)\right| = \left|W(B, C; Y)\right| = |\mathrm{Aut}\, B||\mathrm{Aut}\, C|\left|V(B, C; Y)\right|.$$

By these equalities, we obtain (1.4). This completes the proof. □

2. The Drinfel'd double $\mathcal{D}(\Lambda)$

Following Ringel [8], for finitely generated right Λ-modules $N_1, \ldots, N_t$ and M, we consider the number $\mathbf{F}^M_{N_1,\ldots,N_t}$ of the following filtrations of M:

$$M = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_t = 0,$$

where $U_{i-1}/U_i \cong N_i$ $(1 \leqslant i \leqslant t)$. In particular, we call $\mathbf{F}^M_{N_1 N_2}$ the *Hall* numbers associated to Λ-modules N_1, N_2, and M.

Notation.

$\mathcal{B}(\Lambda)$: the set of isoclasses of Λ-modules.
u_α: the corresponding basis element of $\mathcal{H}(\Lambda)$ for any $\alpha \in \mathcal{B}(\Lambda)$.
V_α: a Λ-module in the class $\alpha \in \mathcal{B}(\Lambda)$.
$\mathbf{F}^\lambda_{\alpha\beta} = \mathbf{F}^L_{MN}$ with M, N, L in $\alpha, \beta, \lambda \in \mathcal{B}(\Lambda)$, respectively.
$\alpha + \beta$: the element in $\mathcal{B}(\Lambda)$ with respect to $V_\alpha \oplus V_\beta$ for $\alpha, \beta \in \mathcal{B}(\Lambda)$.
$a_\alpha = |\mathrm{Aut}_\Lambda(V_\alpha)|$ for $\alpha \in \mathcal{B}(\Lambda)$.
$K_\alpha = K_{\dim V_\alpha}$ for $\alpha \in \mathcal{B}(\Lambda)$, the element of the free Abelian group $\mathbf{Z}[\mathbf{I}]$, where $\mathbf{I}$ is the index set of isoclasses of simple Λ-modules.

We have the *twisted* Ringel–Hall algebra $\mathcal{H}_*(\Lambda)$ over $\mathbf{Q}(v)$ with multiplication given by (cf. [9])

$$u_\alpha u_\beta = v^{\langle\alpha,\beta\rangle} \sum_\gamma \mathbf{F}^\gamma_{\alpha\beta} u_\gamma \quad \text{for } \alpha, \beta \in \mathcal{B}(\Lambda),$$

where $v = \sqrt{|k|}$ and $\langle\alpha, \beta\rangle$ is Euler form of Λ given by

$$\langle\alpha, \beta\rangle = \dim_k \mathrm{Hom}_\Lambda(V_\alpha, V_\beta) - \dim_k \mathrm{Ext}^1_\Lambda(V_\alpha, V_\beta). \tag{2.1}$$

Let $\mathcal{H}_*^{\pm}(\Lambda)$ be the free $\mathbf{Q}(v)$-modules with the bases $\{K_\alpha u_\lambda^{\pm} \mid \alpha \in \mathbf{Z}[\mathbf{I}],\ \lambda \in \mathcal{B}(\Lambda)\}$. By Green–Ringel theory [9], $\mathcal{H}_*^{\pm}(\Lambda)$ are Hopf algebras (Theorem 4.5 in [11]). We have a bilinear form $\varphi : \mathcal{H}_*^{+}(\Lambda) \times \mathcal{H}_*^{-}(\Lambda) \to \mathbf{Q}(v)$ given by

$$\varphi\big(K_\alpha u_\beta^+, K_{\alpha'} u_{\beta'}^-\big) = v^{-(\alpha,\alpha')-(\beta,\alpha')+(\alpha,\beta')} a_\beta^{-1} \delta_{\beta\beta'}.$$

This definition is slightly different from that in Section 5.2 of [11]; here we omit the factor $|u_\beta|$ to simplify computations in our setting. By the proof of Proposition 5.2 in [11], the above φ is a skew-Hopf pair. So we get the Drinfel'd double with respect to φ. The ideal of $\mathcal{D}_0(\Lambda)$ generated by $K_\alpha \otimes K_{-\alpha} - 1$ for any α in $\mathbf{Z}[\mathbf{I}]$ is a Hopf ideal. The corresponding quotient is the *reduced Drinfel'd double* of Λ and denote it by $\mathcal{D}(\Lambda)$. We refer the explicit Hopf structure of $\mathcal{D}(\Lambda)$ to [11].

For brevity, we shall write $x^{\pm}y^{\pm}$ for elements of the form $x^{\pm} \otimes y^{\pm}$ in $\mathcal{D}(\Lambda)$. We have the following

Lemma 2.1. *For $\lambda, \lambda' \in \mathcal{B}(\Lambda)$ with $\mathrm{Hom}_\Lambda(V_\lambda, V_{\lambda'}) = 0$, in $\mathcal{D}(\Lambda)$ we have*

$$u_{\lambda'}^- u_\lambda^+ - u_\lambda^+ u_{\lambda'}^- = \sum_{\substack{\beta,\beta',\gamma \in \mathcal{B}(\Lambda)\\ \gamma \neq 0}} v^{\langle\beta,\gamma\rangle - \langle\beta',\gamma\rangle} \mathbf{F}_{\beta\gamma}^{\lambda} \mathbf{F}_{\gamma\beta'}^{\lambda'} a_\beta a_\gamma a_{\beta'} a_\lambda^{-1} a_{\lambda'}^{-1} u_\beta^+ u_{\beta'}^- K_\gamma.$$

Proof. Completely similar to Proposition 5.5 in [11], we have

$$\begin{aligned}
u_{\lambda'}^- u_\lambda^+ - u_\lambda^+ u_{\lambda'}^- = {} & \sum_{\substack{\beta,\beta',\gamma \in \mathcal{B}(\Lambda)\\ \gamma \neq 0}} v^{\langle\beta,\gamma\rangle - \langle\beta',\gamma\rangle} \mathbf{F}_{\beta\gamma}^{\lambda} \mathbf{F}_{\gamma\beta'}^{\lambda'} a_\beta a_\gamma a_{\beta'} a_\lambda^{-1} a_{\lambda'}^{-1} u_\beta^+ u_{\beta'}^- K_\gamma \\
& + \sum_{\substack{\beta,\gamma,\beta',\alpha,\gamma' \in \mathcal{B}(\Lambda)\\ \alpha,\gamma' \neq 0}} v^{\langle\beta,\gamma\rangle - \langle\beta,\alpha\rangle + \langle\beta',\alpha\rangle - \langle\beta',\gamma\rangle} \\
& \qquad \times \mathbf{F}_{\alpha\beta\gamma}^{\lambda} \mathbf{F}_{\gamma\beta'\gamma'}^{\lambda'} a_\beta a_\gamma a_{\beta'} a_\lambda^{-1} a_{\lambda'}^{-1} \\
& \qquad \times \sum_{m \geqslant 1} (-1)^m \sum_{\lambda_1,\dots,\lambda_m \in \mathcal{B}(\Lambda)-\{0\}} a_{\lambda_1} \dots a_{\lambda_m} \\
& \qquad \times \mathbf{F}_{\lambda_1\dots\lambda_m}^{\gamma'} \mathbf{F}_{\lambda_m\dots\lambda_1}^{\alpha} u_\beta^+ u_{\beta'}^- K_\gamma K_{-\gamma'}.
\end{aligned}$$

Now assume that $\mathbf{F}_{\alpha\beta\gamma}^{\lambda} \mathbf{F}_{\gamma\beta'\gamma'}^{\lambda'} \neq 0$ for some nonzero α and γ'. So there are short exact sequences of the form $0 \to V_\delta \to V_\lambda \xrightarrow{f_1} V_\alpha \to 0$ and $0 \to V_{\gamma'} \xrightarrow{f_4} V_{\lambda'} \to V_{\delta'} \to 0$. For any $m \geqslant 1$, assume that $\mathbf{F}_{\lambda_1\dots\lambda_m}^{\gamma'} \mathbf{F}_{\lambda_m\dots\lambda_1}^{\alpha} \neq 0$ for some nonzero λ_i. Then there is a nonzero module V_{λ_m} which is a submodule of $V_{\gamma'}$ and a factor module of V_α at the same time. Thus, we have a chain of morphisms: $V_\lambda \xrightarrow{f_1} V_\alpha \xrightarrow{f_2} V_{\lambda_m} \xrightarrow{f_3} V_{\gamma'} \xrightarrow{f_4} V_{\lambda'}$, where f_2 is an epimorphism and f_3 is a monomorphism, respectively. Since $\mathrm{Hom}_\Lambda(V_\lambda, V_{\lambda'}) = 0$, we have that $f_1 f_2 = 0$, and hence $V_{\lambda_m} = 0$, a contradiction. This completes the proof. $\square$

We shall use an equivalent presentation of $\mathcal{D}(\Lambda)$ via generators and definition relations. Similar to that in [10,12], $\mathcal{D}(\Lambda)$ is generated by K_γ, $u_\alpha^{\pm}$ with γ in $\mathbf{Z}[\mathbf{I}]$, and α in $\mathcal{B}(\Lambda)$, respectively, subject to (2.2)–(2.4)

$$K_0 = u_0^{\pm} = 1, \qquad K_\gamma K_\delta = K_{\gamma+\delta}, \tag{2.2}$$

$$u_\alpha^{\pm} u_\beta^{\pm} = v^{\langle\alpha,\beta\rangle} \sum \mathbf{F}_{\alpha\beta}^{\lambda} u_\lambda^{\pm}, \qquad K_\alpha u_\beta^{\pm} = v^{\pm(\alpha,\beta)} u_\beta^{\pm} K_\alpha, \tag{2.3}$$

$$\sum_{\alpha,\beta,\alpha'} v^{\langle\alpha,\alpha'\rangle - \langle\beta,\alpha'\rangle} a_\alpha a_\beta a_{\alpha'} \mathbf{F}_{\alpha'\beta}^{\lambda} \mathbf{F}_{\alpha\alpha'}^{\lambda'} u_\alpha^- u_\beta^+ K_{-\alpha'}$$
$$= \sum_{\alpha,\beta,\beta'} v^{\langle\alpha,\beta'\rangle - \langle\beta,\beta'\rangle} a_\alpha a_\beta a_{\beta'} \mathbf{F}_{\alpha\beta'}^{\lambda} \mathbf{F}_{\beta'\beta}^{\lambda'} u_\alpha^+ u_\beta^- K_{\beta'} \tag{2.4}$$

for λ, λ' in $\mathcal{B}(\Lambda)$, where $(,)$ is the symmetrization of Euler form given by (2.1). It is easy to see that there is an automorphism ω of $\mathcal{D}(\Lambda)$ given by

$$\omega : u_\alpha^{\pm} \mapsto v^{-\langle\alpha,\alpha\rangle} u_\alpha^{\mp} K_{\pm\alpha}, \qquad K_\gamma \mapsto K_{-\gamma} \tag{2.5}$$

for any $\alpha \in \mathcal{B}(\Lambda)$ and $\gamma \in \mathbf{Z}[\mathbf{I}]$.

3. The embedding

Let T be a tilting module of Λ and $\Gamma = \operatorname{End}_\Lambda T$ be the corresponding tilted algebra [1]. Functors $\operatorname{Hom}_\Lambda(T, -)$ and $- \otimes_\Gamma T$ induce mutually inverse equivalences between the following full subcategories of $\operatorname{mod} \Lambda$ and $\operatorname{mod} \Gamma$:

$$\mathcal{T}(T) = \{M \mid \operatorname{Ext}_\Lambda^1(T, M) = 0\} \quad \text{and} \quad \mathcal{Y}(T) = \{N \mid \operatorname{Tor}_1^\Gamma(N, T) = 0\}$$

while functors $\operatorname{Ext}_\Lambda^1(T, -)$ and $\operatorname{Tor}_1^\Gamma(-, T)$ induce mutually inverse equivalences between the following full subcategories of $\operatorname{mod} \Lambda$ and $\operatorname{mod} \Gamma$:

$$\mathcal{F}(T) = \{M \mid \operatorname{Hom}_\Lambda(T, M) = 0\} \quad \text{and} \quad \mathcal{X}(T) = \{N \mid N \otimes_\Gamma T = 0\}.$$

Notation.

$\mathcal{B}(\Lambda, \mathcal{T})$: the set of isoclasses of Λ-modules in $\mathcal{T}(T)$.
$\mathcal{B}(\Lambda, \mathcal{F})$: the set of isoclasses of Λ-modules in $\mathcal{F}(T)$.
$\mathcal{B}(\Gamma, \mathcal{X})$: the set of isoclasses of Γ-modules in $\mathcal{X}(T)$.
$\mathcal{B}(\Gamma, \mathcal{Y})$: the set of isoclasses of Γ-modules in $\mathcal{Y}(T)$.
$T(\alpha) \in \mathcal{B}(\Gamma, \mathcal{Y})$: the isoclass of $\operatorname{Hom}_\Lambda(T, V_\alpha)$ for $\alpha \in \mathcal{B}(\Lambda, \mathcal{T})$.
$T(\alpha) \in \mathcal{B}(\Gamma, \mathcal{X})$: the isoclass of $\operatorname{Ext}_\Lambda^1(T, V_\alpha)$ for $\alpha \in \mathcal{B}(\Lambda, \mathcal{T})$.
$T^-(\alpha) \in \mathcal{B}(\Lambda, \mathcal{F})$: the isoclass of $\operatorname{Tor}_1^\Gamma(V_\alpha, T)$ for $\alpha \in \mathcal{B}(\Gamma, \mathcal{X})$.
$T^-(\alpha) \in \mathcal{B}(\Lambda, \mathcal{T})$: the isoclass of $V_\alpha \otimes_\Gamma T$ for $\alpha \in \mathcal{B}(\Gamma, \mathcal{Y})$.

Let $K_0(\Lambda)$ and $K_0(\Gamma)$ be the *Grothendieck groups* of Λ and Γ, respectively. By Theorem 2.5 in [1], there is an isomorphism $\mathrm{f}: K_0(\Lambda) \to K_0(\Gamma)$ given by

$$\mathrm{f}(\dim M) = \dim \mathrm{Hom}_\Lambda(T, M) - \dim \mathrm{Ext}^1_\Lambda(T, M) \quad \text{for } M \in \mathrm{mod}\,\Lambda. \tag{3.1}$$

Consider Euler forms $\langle,\rangle_\Lambda$ and $\langle,\rangle_\Gamma$ on $K_0(\Lambda)$ and $K_0(\Gamma)$, respectively. For M, N in $\mathrm{mod}\,\Lambda$,

$$\langle \dim M, \dim N\rangle_\Lambda = \dim_k \mathrm{Hom}_\Lambda(M, N) - \dim_k \mathrm{Ext}^1_\Lambda(M, N),$$

and for M, N in $\mathrm{mod}\,\Gamma$,

$$\langle \dim M, \dim N\rangle_\Gamma = \dim_k \mathrm{Hom}_\Gamma(M, N) - \dim_k \mathrm{Ext}^1_\Gamma(M, N) + \dim_k \mathrm{Ext}^2_\Gamma(M, N).$$

By Theorem 2.2 and Proposition 2.7 in [1], we have that

$$\langle \dim M, \dim N\rangle_\Lambda = \big\langle \mathrm{f}(\dim M), \mathrm{f}(\dim N)\big\rangle_\Gamma \quad \text{for all } M, N \in \mathrm{mod}\,\Lambda.$$

Since $\langle,\rangle_\Lambda$ depends only on dimension vectors, so does $\langle,\rangle_\Gamma$. Moreover,

$$\begin{aligned}
\langle \dim M, \dim N\rangle_\Lambda &= \langle \dim M', \dim N'\rangle_\Gamma \quad \text{for } M, N \in \mathcal{T}(T),\\
\langle \dim M, \dim N\rangle_\Lambda &= \langle \dim M', \dim N'\rangle_\Gamma \quad \text{for } M, N \in \mathcal{F}(T),\\
\langle \dim M, \dim N\rangle_\Lambda &= -\langle \dim M', \dim N'\rangle_\Gamma \quad \text{for } M \in \mathcal{T}(T),\ N \in \mathcal{F}(T),\\
\langle \dim M, \dim N\rangle_\Lambda &= -\langle \dim M', \dim N'\rangle_\Gamma \quad \text{for } M \in \mathcal{F}(T),\ N \in \mathcal{T}(T),
\end{aligned}$$

where

$$\begin{aligned}
M' &= \mathrm{Hom}_\Lambda(T, M) \quad \text{if } M \in \mathcal{T}(T),\\
&= \mathrm{Ext}^1_\Lambda(T, M) \quad \text{if } M \in \mathcal{F}(T),
\end{aligned}$$

and N' is given by the same way.

Let $\mathcal{H}_*(\Gamma)$ be the free Abelian group over $\mathbf{Q}(v)$ with $\mathcal{B}(\Gamma)$ being its basis. Again we denote $\langle \dim V_\alpha, \dim V_\beta\rangle_\Gamma$ by $\langle \alpha, \beta\rangle_\Gamma$ for α, β in $\mathcal{B}(\Gamma)$. Define a multiplication on $\mathcal{H}_*(\Gamma)$ as following

$$u_\alpha u_\beta = v^{\langle \alpha,\beta\rangle_\Gamma} \sum_\gamma \mathbf{F}^\gamma_{\alpha\beta} u_\gamma \tag{3.2}$$

for any $\alpha, \beta \in \mathcal{B}(\Gamma)$, where $\mathbf{F}^\gamma_{\alpha\beta}$ is Hall number. By the definition of Hall numbers, it is easy to see that $\mathcal{H}_*(\Gamma)$ becomes an associative algebra over $\mathbf{Q}(v)$ with the multiplication given by (3.2).

From now on we omit subscripts of Euler forms $\langle,\rangle_\Lambda$ and $\langle,\rangle_\Gamma$, which should not cause ambiguity by the context. At first we have the following

Lemma 3.1. *For $\lambda, \alpha \in \mathcal{B}(\Lambda, \mathcal{T})$ and $\lambda', \beta' \in \mathcal{B}(\Lambda, \mathcal{F})$, we have that*

$$\mathbf{F}^{T(\alpha)+T(\beta')}_{T(\lambda')T(\lambda)} a_{T(\lambda)} a_{T(\lambda')} = v^{-2\langle\alpha,\beta'\rangle} \sum_{\beta\in\mathcal{B}(\Lambda)} \mathbf{F}^{\lambda}_{\alpha\beta}\mathbf{F}^{\lambda'}_{\beta\beta'} a_\alpha a_\beta a_{\beta'}. \tag{3.3}$$

Proof. Note that the subcategories $\mathcal{T}(T)$ and $\mathcal{F}(T)$ in $\mathrm{mod}\,\Lambda$ induced by a tilting module T of Λ satisfy the condition of Theorem 1.1. We interpret (1.4) in terms of Hall numbers in $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$, respectively. There is a natural equivalence between the derived categories $D^b(\Lambda)$ and $D^b(\Gamma)$ via tilting functors [5]. Thus, there is an embedding of $\mathrm{mod}\,\Gamma$ into $D^b(\Lambda)$ such that $\mathcal{T}(T) \simeq \mathcal{Y}(T)$ and $\mathbf{T}\mathcal{F}(T) \simeq \mathcal{X}(T)$ as full subcategories of $D^b(\Lambda)$. All these equivalences are given by tilting functors. It is known that the Hall numbers $\mathbf{F}^B_{CA}$ equals to the cardinality of $V(A, C; B)$ when A, B, C lie in a module category of Λ or Γ [6], and the result follows. □

Dually, we have the following

Lemma 3.2. *Assume that the tilted algebra Γ is hereditary. Keep notation as above. For $\lambda, \alpha \in \mathcal{B}(\Gamma, \mathcal{Y})$ and $\lambda', \beta' \in \mathcal{B}(\Gamma, \mathcal{X})$, we have that*

$$\mathbf{F}^{T^-(\alpha)+T^-(\beta')}_{T^-(\lambda')T^-(\lambda)} a_{T^-(\lambda)} a_{T^-(\lambda')} = v^{-2\langle\alpha,\beta'\rangle} \sum_{\beta\in\mathcal{B}(\Gamma)} \mathbf{F}^{\lambda}_{\alpha\beta}\mathbf{F}^{\lambda'}_{\beta\beta'} a_\alpha a_\beta a_{\beta'}. \tag{3.4}$$

In next section we shall use the following

Lemma 3.3. *Assume that the tilted algebra Γ is hereditary. Keep notation as before. For λ', α', y in $\mathcal{B}(\Lambda, \mathcal{T})$ and x in $\mathcal{B}(\Lambda, \mathcal{F})$, we have that*

$$\mathbf{F}^{T(\lambda')}_{T(y)+T(x)T(\alpha')} a_{T(\alpha')} a_{T(y)+T(x)} = v^{-2\langle y,x\rangle} \sum_{\gamma\in\mathcal{B}(\Lambda,\mathcal{T})} \mathbf{F}^{\alpha'}_{\gamma x}\mathbf{F}^{\lambda'}_{y\gamma} a_\gamma a_x a_y. \tag{3.5}$$

Similarly, for any λ, α', x in $\mathcal{B}(\Lambda, \mathcal{F})$ and y in $\mathcal{B}(\Lambda, \mathcal{T})$, we have that

$$\mathbf{F}^{T(\lambda)}_{T(\alpha')T(y)+T(x)} a_{T(\alpha')} a_{T(y)+T(x)} = v^{-2\langle y,x\rangle} \sum_{\gamma\in\mathcal{B}(\Lambda,\mathcal{F})} \mathbf{F}^{\lambda}_{\gamma x}\mathbf{F}^{\alpha'}_{y\gamma} a_\gamma a_x a_y. \tag{3.6}$$

Proof. We show (3.5) since the proof of (3.6) is completely similar. For brevity, set $f^* = \mathrm{Hom}_\Lambda(T, f)$ for morphism $f: M \to N$ in $\mathrm{mod}\,\Lambda$. For $\lambda', \alpha', y \in \mathcal{B}(\Lambda, \mathcal{T})$ and $x \in \mathcal{B}(\Lambda, \mathcal{F})$, fix Λ-modules $V_{\lambda'}$, $V_{\alpha'}$, V_y, and V_x. Set

$$\mathrm{E}_1 = \left\{(s, t_1, t_2) \;\middle|\; 0 \to V_{T(\alpha')} \xrightarrow{s} V_{T(\lambda')} \xrightarrow{(t_1,t_2)} V_{T(y)} \oplus V_{T(x)} \to 0 \text{ is exact}\right\} \quad \text{and}$$

$$\mathrm{E}_2 = \left\{(h, f, g) \;\middle|\; 0 \to V_x \xrightarrow{h} V_{\alpha'} \xrightarrow{f} V_{\lambda'} \xrightarrow{g} V_y \to 0 \text{ is exact}\right\}.$$

Then, by definition of Hall numbers, the left-hand side of (3.5) is $|\mathrm{E}_1|$ while the right-hand side is $v^{-2\langle y,x\rangle}|\mathrm{E}_2|$. There is an action of $\operatorname{Aut} V_x$ on E_2 given by

$$\alpha\big((h,f,g)\big)=(\alpha^{-1}h,f,g)\quad\text{for } \alpha\in\operatorname{Aut} V_x \text{ and } (h,f,g)\in\mathrm{E}_2.$$

Denote the orbit space by $\mathrm{E}_2^\vee$. Since this action is free, we have $|\mathrm{E}_2|=a_x|\mathrm{E}_2^\vee|$.

Now for any $(s,t_1,t_2)\in\mathrm{E}_1$, applying $-\otimes_\Gamma T$ to the short exact sequence $0\to V_{T(\alpha')}\xrightarrow{s} V_{T(\lambda')}\xrightarrow{(t_1,t_2)} V_{T(y)}\oplus V_{T(x)}\to 0$ we obtain an exact sequence

$$0\to V_x\xrightarrow{h} V_{\alpha'}\xrightarrow{s\otimes_\Gamma T} V_{\lambda'}\xrightarrow{t_1\otimes_\Gamma T} V_y\to 0$$

for some morphism h, where $s\otimes_\Gamma T$ and $t_1\otimes_\Gamma T$ are uniquely determined by s and t_1, respectively, since $-\otimes_\Gamma T$ is an equivalence between $\mathcal{Y}(T)$ and $\mathcal{T}(T)$. So we obtain a unique element $(h,s\otimes_\Gamma T,t_1\otimes_\Gamma T)^\vee\in\mathrm{E}_2^\vee$ and we define a map $\pi:\mathrm{E}_1\to\mathrm{E}_2^\vee$ by $(s,t_1,t_2)\mapsto(h,s\otimes_\Gamma T,t_1\otimes_\Gamma T)^\vee$. We claim that π is surjective. In fact, for any exact sequence

$$0\to V_x\xrightarrow{h} V_{\alpha'}\xrightarrow{f} V_{\lambda'}\xrightarrow{g} V_y\to 0,$$

set $V_\gamma=\operatorname{Im} f\in\mathcal{T}(T)$. Then there are two short exact sequences

$$0\to V_x\xrightarrow{h} V_{\alpha'}\xrightarrow{f} V_\gamma\to 0\quad\text{and}\quad 0\to V_\gamma\xrightarrow{i} V_{\lambda'}\xrightarrow{g} V_y\to 0,$$

where i is the canonical embedding. Applying $\operatorname{Hom}_\Lambda(T,-)$ to these exact sequences we obtain two short exact sequences

$$0\to V_{T(\alpha')}\xrightarrow{f^*} V_{T(\gamma)}\xrightarrow{\delta} V_{T(x)}\to 0\quad\text{and}\quad 0\to V_{T(\gamma)}\xrightarrow{i^*} V_{T(\lambda')}\xrightarrow{g^*} V_{T(y)}\to 0$$

for some morphisms δ. By push-out of $\delta:V_{T(\gamma)}\to V_{T(x)}$, we get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & V_{T(\alpha')} \arrow[r, equal] \arrow[d, "f^*"] & V_{T(\alpha')} \arrow[d, "f^*"] & & \\
0 \arrow[r] & V_{T(\gamma)} \arrow[r, "i^*"] \arrow[d, "\delta"] & V_{T(\lambda')} \arrow[r, "g^*"] \arrow[d, "t"] & V_{T(y)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V_{T(x)} \arrow[r, "k"] \arrow[d] & M \arrow[r, "l"] \arrow[d] & V_{T(y)} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd},$$

where k, l, and M are obtained by the standard push-out construction. Since $(\mathcal{X}(T), \mathcal{Y}(T))$ splits, the last row splits and hence there is an isomorphism $\delta_1 : M \to V_{T(y)} \oplus V_{T(x)}$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & V_{T(x)} \arrow[r, "k"] \arrow[d, equal] & M \arrow[r, "l"] \arrow[d, "\delta_1"] & V_{T(y)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V_{T(x)} \arrow[r, "(0,1)"] & V_{T(y)} \oplus V_{T(x)} \arrow[r, "(0,1)^{\mathrm{t}}"] & V_{T(y)} \arrow[r] & 0
\end{tikzcd}.$$

Thus, we obtain a commutative diagram as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & V_{T(\alpha')} \arrow[r, equal] \arrow[d, "f^*"] & V_{T(\alpha')} \arrow[d, "f^*"] & & \\
0 \arrow[r] & V_{T(\gamma)} \arrow[r, "i^*"] \arrow[d, "\delta"] & V_{T(\lambda')} \arrow[r, "g^*"] \arrow[d, "(t_1,t_2)"] & V_{T(y)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V_{T(x)} \arrow[r, "(0,1)"] \arrow[d] & V_{T(y)} \oplus V_{T(x)} \arrow[r, "(l,0)^{\mathrm{t}}"] \arrow[d] & V_{T(y)} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}.$$

So $t_1 = g^*$. By the definition of π, we have that $\pi((f^*, t_1, t_2)) = (h, f, g)^\vee$, which means that π is surjective.

Note that for any $(h, f, g)^\vee \in \mathrm{E}_2^\vee$, we have that

$$\begin{aligned}
&\pi^{-1}\big((h, f, g)^\vee\big) \\
&\quad = \big\{(f^*, g^*, t_2) \bigm| 0 \to V_{T(\alpha')} \xrightarrow{f^*} V_{T(\lambda')} \xrightarrow{(g^*,t_2)} V_{T(y)} \oplus V_{T(x)} \to 0 \text{ is exact}\big\}.
\end{aligned}$$

Since g^* is an epimorphism, we have that

$$\big|\pi^{-1}\big((h, f, g)^\vee\big)\big| = \left|\left\{ \begin{pmatrix} 1 & h_1 \\ 0 & \alpha \end{pmatrix} \,\middle|\, \alpha \in \operatorname{Aut} V_{T(x)} \text{ and } h_1 \in \operatorname{Hom}_\Gamma\big(V_{T(y)}, V_{T(x)}\big)\right\}\right|$$

and hence $|\pi^{-1}((h, f, g)^\vee)| = a_x |\operatorname{Hom}_\Gamma(V_{T(y)}, V_{T(x)})|$. Since Γ is hereditary, we have that $\dim_k \operatorname{Hom}_\Gamma(V_{T(y)}, V_{T(x)}) = \langle T(y), T(x)\rangle_\Gamma = -\langle y, x\rangle$ and hence

$$
\begin{aligned}
|\mathrm{E}_1| &= a_x \left|\mathrm{Hom}_\Gamma\left(V_{T(y)}, V_{T(x)}\right)\right|\left|\mathrm{E}_2^\vee\right| = a_x \left|\mathrm{Hom}_\Gamma\left(V_{T(y)}, V_{T(x)}\right)\right| a_x^{-1}|\mathrm{E}_2| \\
&= \left|\mathrm{E}_2^\vee\right|\left|\mathrm{Hom}_\Gamma\left(V_{T(y)}, V_{T(x)}\right)\right| = |k|^{-\langle y,x\rangle}|\mathrm{E}_2| = v^{-2\langle y,x\rangle}|\mathrm{E}_2|.
\end{aligned}
$$

This completes the proof. □

Since the torsion theory $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting, for any basis element u_γ with $\gamma \in \mathcal{B}(\Gamma)$, we always have $\gamma = T(\alpha) + T(\beta')$ for some $\alpha \in \mathcal{B}(\Lambda, \mathcal{T})$ and $\beta' \in \mathcal{B}(\Lambda, \mathcal{F})$. So, we can define a $\mathbf{Q}(v)$-linear map $\Psi : \mathcal{H}_*(\Gamma) \to \mathcal{D}(\Lambda)$ by

$$\Psi : u_\gamma \mapsto v^{\langle\alpha,\beta'\rangle - \langle\beta',\beta'\rangle} u_\alpha^+ K_{-\beta'} u_{\beta'}^-, \tag{3.7}$$

where $\gamma = T(\alpha) + T(\beta')$, $\alpha \in \mathcal{B}(\Lambda, \mathcal{T})$, and $\beta' \in \mathcal{B}(\Lambda, \mathcal{F})$. Our main result in this section is the following

Theorem 3.1. *The map Ψ given by* (3.7) *is a monomorphism of $\mathbf{Q}(v)$-algebras.*

At first, it is easy to see that we have the following

Lemma 3.4. *The map Ψ given by* (3.7) *is an algebra morphism if and only if the following conditions are satisfied*:

(1) $\Psi(u_{\gamma_1} u_{\gamma_2}) = \Psi(u_{\gamma_1})\Psi(u_{\gamma_2})$ *for* $\gamma_1, \gamma_2 \in \mathcal{B}(\Gamma, \mathcal{Y})$;
(2) $\Psi(u_{\gamma_1} u_{\gamma_2}) = \Psi(u_{\gamma_1})\Psi(u_{\gamma_2})$ *for* $\gamma_1, \gamma_2 \in \mathcal{B}(\Gamma, \mathcal{X})$;
(3) $\Psi(u_{\gamma_1} u_{\gamma_2}) = \Psi(u_{\gamma_1})\Psi(u_{\gamma_2})$ *for* $\gamma_1 \in \mathcal{B}(\Gamma, \mathcal{Y})$ *and* $\gamma_2 \in \mathcal{B}(\Gamma, \mathcal{X})$;
(4) $\Psi(u_{\lambda_0'} u_{\lambda_0}) = \Psi(u_{\lambda_0'})\Psi(u_{\lambda_0})$ *for* $\lambda_0 \in \mathcal{B}(\Gamma, \mathcal{Y})$ *and* $\lambda_0' \in \mathcal{B}(\Gamma, \mathcal{X})$.

Proof of Theorem 3.1. By Lemma 3.4, we should check (1)–(4). At first, (1) is obvious. Similarly, we can verify (2). Assume that $T(\gamma_1) \in \mathcal{B}(\Gamma, \mathcal{Y})$ and $T(\gamma_2) \in \mathcal{B}(\Gamma, \mathcal{X})$. By (3.7), we have that

$$\Psi(u_{T(\gamma_1)})\Psi(u_{T(\gamma_2)}) = v^{-\langle\gamma_2,\gamma_2\rangle} u_{\gamma_1}^+ K_{-\gamma_2} u_{\gamma_2}^-.$$

On the other hand, by (3.7), we have that

$$\Psi(u_{T(\gamma_1)} u_{T(\gamma_2)}) = v^{-\langle\gamma_1,\gamma_2\rangle}\Psi(u_{T(\gamma_1)+T(\gamma_2)}) = v^{-\langle\gamma_2,\gamma_2\rangle} u_{\gamma_1}^+ K_{-\gamma_2} u_{\gamma_2}^-,$$

which verifies (3). Now we assume that $\lambda_0' = T(\lambda') \in \mathcal{B}(\Gamma, \mathcal{X})$ and $\lambda_0 = T(\lambda) \in \mathcal{B}(\Gamma, \mathcal{Y})$. Since $\lambda' \in \mathcal{B}(\Lambda, \mathcal{F})$ and $\lambda \in \mathcal{B}(\Lambda, \mathcal{T})$ respectively, we have $\mathrm{Hom}_\Lambda(V_\lambda, V_{\lambda'}) = 0$. Recalling Lemma 2.1 and (3.7) we have that

$$
\begin{aligned}
\Psi(u_{\lambda_0'})\Psi(u_{\lambda_0}) &= v^{-\langle\lambda',\lambda'\rangle} K_{-\lambda'} u_{\lambda'}^- u_\lambda^+ \\
&= \sum_{\alpha,\beta,\beta' \in \mathcal{B}(\Lambda)} v^{-\langle\lambda',\lambda'\rangle + \langle\alpha,\beta\rangle + \langle\beta,\beta'\rangle - \langle\lambda',\alpha\rangle} a_\alpha a_\beta a_{\beta'} a_\lambda^{-1} a_{\lambda'}^{-1} \\
&\qquad \times \mathbf{F}_{\alpha\beta}^{\lambda} \mathbf{F}_{\beta\beta'}^{\lambda'} u_\alpha^+ K_{-\beta'} u_{\beta'}^-.
\end{aligned}
$$

On the other hand, by (3.7), we have that

$$\Psi(u_{\lambda_0'}u_{\lambda_0}) = \sum_{\substack{\alpha\in\mathcal{B}(\Lambda,\mathcal{T})\\ \beta'\in\mathcal{B}(\Lambda,\mathcal{F})}} v^{-\langle\lambda',\lambda\rangle}\mathbf{F}_{T(\lambda')T(\lambda)}^{T(\alpha)+T(\beta')}\Psi(u_{T(\alpha)+T(\beta')})$$
$$= \sum_{\substack{\alpha\in\mathcal{B}(\Lambda,\mathcal{T})\\ \beta'\in\mathcal{B}(\Lambda,\mathcal{F})}} v^{-\langle\lambda',\lambda\rangle-\langle\beta',\beta'\rangle+\langle\alpha,\beta'\rangle}\mathbf{F}_{T(\lambda')T(\lambda)}^{T(\alpha)+T(\beta')}u_\alpha^+K_{-\beta'}u_{\beta'}^-.$$

By (3.3), $\mathbf{F}_{T(\lambda')T(\lambda)}^{T(\alpha)+T(\beta')}a_{T(\lambda)}a_{T(\lambda')} = v^{-2\langle\alpha,\beta'\rangle}\sum_{\beta\in\mathcal{B}(\Lambda)}\mathbf{F}_{\alpha\beta}^{\lambda}\mathbf{F}_{\beta\beta'}^{\lambda'}a_\alpha a_\beta a_{\beta'}$. Moreover, we may assume that

$$\dim V_\lambda = \dim V_\alpha + \dim V_\beta, \qquad \dim V_{\lambda'} = \dim V_\beta + \dim V_{\beta'}.$$

Thus, $\dim V_{\lambda'} = \dim V_\lambda + \dim V_{\beta'} - \dim V_\alpha$. So we have that

$$-\langle\lambda',\lambda'\rangle + \langle\alpha,\beta\rangle + \langle\beta,\beta'\rangle - (\lambda',\alpha) = -\langle\lambda',\lambda\rangle - \langle\beta',\beta'\rangle - \langle\alpha,\beta'\rangle.$$

It follows that $\Psi(u_{\lambda_0'}u_{\lambda_0}) = \Psi(u_{\lambda_0'})\Psi(u_{\lambda_0})$, which means that (4) in Lemma 3.4 holds. This completes the proof. □

4. The isomorphism

In this section we assume that T is a slice module of Λ and hence the tilted algebra Γ is also hereditary [1]. Then we have Drinfel'd doubles $\mathcal{D}(\Lambda)$ and $\mathcal{D}(\Gamma)$. Keep notation as before. For brevity, we write the Euler characteristic $\langle,\rangle_\Gamma$ as $\langle,\rangle$ and its symmetrization as $(,)$. Similar to the presentation (2.2)–(2.4) of $\mathcal{D}(\Lambda)$, $\mathcal{D}(\Gamma)$ is generated by K_{γ_0}, $u_{\alpha_0}^\pm$ with γ_0 in $K_0(\Gamma)$, and α_0 in $\mathcal{B}(\Gamma)$, respectively, subject to (4.1)–(4.3)

$$K_0 = u_0^\pm = 1, \qquad K_{\gamma_0}K_{\delta_0} = K_{\gamma_0+\delta_0}, \tag{4.1}$$

$$u_{\alpha_0}^\pm u_{\beta_0}^\pm = v^{\langle\alpha_0,\beta_0\rangle}\sum\mathbf{F}_{\alpha_0\beta_0}^{\lambda_0}u_{\lambda_0}^\pm, \qquad K_{\alpha_0}u_{\beta_0}^\pm = v^{\pm(\alpha_0,\beta_0)}u_{\beta_0}^\pm K_{\alpha_0}, \tag{4.2}$$

$$\sum_{\alpha_0,\beta_0,\alpha_0'} v^{\langle\alpha_0,\alpha_0'\rangle-\langle\beta_0,\alpha_0'\rangle}a_{\alpha_0}a_{\beta_0}a_{\alpha_0'}\mathbf{F}_{\alpha_0'\beta_0}^{\lambda_0}\mathbf{F}_{\alpha_0\alpha_0'}^{\lambda_0'}u_{\alpha_0}^-u_{\beta_0}^+K_{-\alpha_0'}$$
$$= \sum_{\alpha_0,\beta_0,\beta_0'} v^{\langle\alpha_0,\beta_0'\rangle-\langle\beta_0,\beta_0'\rangle}a_{\alpha_0}a_{\beta_0}a_{\beta_0'}\mathbf{F}_{\alpha_0\beta_0'}^{\lambda_0}\mathbf{F}_{\beta_0'\beta_0}^{\lambda_0'}u_{\alpha_0}^+u_{\beta_0}^-K_{\beta_0'} \tag{4.3}$$

for λ_0, λ_0' in $\mathcal{B}(\Gamma)$.

Now we state the main result of this section as following

Theorem 4.1. *There is a* $\mathbf{Q}(v)$*-algebra isomorphism* $\Psi : \mathcal{D}(\Gamma) \to \mathcal{D}(\Lambda)$ *given by*

$$u_{\gamma_0}^{\pm} \mapsto v^{\langle \alpha, \beta' \rangle - \langle \beta', \beta' \rangle} u_{\alpha}^{\pm} K_{\mp \beta'} u_{\beta'}^{\mp}, \qquad K_{\lambda_0} \mapsto K_{\lambda} \in K_0(\Lambda), \tag{4.4}$$

where $\gamma_0 = T(\alpha) + T(\beta')$ *for* $\alpha \in \mathcal{B}(\Lambda, \mathcal{T})$, $\beta' \in \mathcal{B}(\Lambda, \mathcal{F})$, *and, under the action of* f *given by* (3.1), $\mathrm{f}(\lambda) = \lambda_0 \in K_0(\Gamma)$.

Note that for any α in $\mathcal{B}(\Lambda)$, we have the following facts:

$$K_{\mathrm{f}(\alpha)} = K_{T(\alpha)} \quad \text{if } \alpha \in \mathcal{B}(\Lambda, \mathcal{T}),$$
$$K_{\mathrm{f}(\alpha)} = K_{-T(\alpha)} \quad \text{if } \alpha \in \mathcal{B}(\Lambda, \mathcal{F}).$$

As in [10], we shall show that Ψ respects relations (4.1)–(4.3). The most complicated computation in our proof will be the verification that Ψ respects (4.3). However, thanks to Lemma 3.2 in [10], we can reduce it to some special cases. We prove the following

Lemma 4.1. *Assume that* λ *and* λ' *belong to either* $\mathcal{B}(\Lambda, \mathcal{T})$ *or* $\mathcal{B}(\Lambda, \mathcal{F})$*. In each one of the following cases*

(a) $\lambda_0 = T(\lambda) \in \mathcal{B}(\Gamma, \mathcal{Y})$ *and* $\lambda_0' = T(\lambda') \in \mathcal{B}(\Gamma, \mathcal{X})$,
(b) $\lambda_0 = T(\lambda) \in \mathcal{B}(\Gamma, \mathcal{X})$ *and* $\lambda_0' = T(\lambda') \in \mathcal{B}(\Gamma, \mathcal{Y})$,
(c) $\lambda_0 = T(\lambda)$, $\lambda_0' = T(\lambda') \in \mathcal{B}(\Gamma, \mathcal{Y})$,
(d) $\lambda_0 = T(\lambda)$, $\lambda_0' = T(\lambda') \in \mathcal{B}(\Gamma, \mathcal{X})$,

we have the following equality in $\mathcal{D}(\Lambda)$:

$$\begin{aligned}
&\sum_{\alpha_0, \beta_0, \alpha_0'} v^{\langle \alpha_0, \alpha_0' \rangle - \langle \beta_0, \alpha_0' \rangle} a_{\alpha_0} a_{\beta_0} a_{\alpha_0'} \mathbf{F}_{\alpha_0' \beta_0}^{\lambda_0} \mathbf{F}_{\alpha_0 \alpha_0'}^{\lambda_0'} \Psi\left(u_{\alpha_0}^{-}\right) \Psi\left(u_{\beta_0}^{+}\right) \Psi\left(K_{-\alpha_0'}\right) \\
&\quad = \sum_{\alpha_0, \beta_0, \beta_0'} v^{\langle \alpha_0, \beta_0' \rangle - \langle \beta_0, \beta_0' \rangle} a_{\alpha_0} a_{\beta_0} a_{\beta_0'} \mathbf{F}_{\alpha_0 \beta_0'}^{\lambda_0} \mathbf{F}_{\beta_0' \beta_0}^{\lambda_0'} \Psi\left(u_{\alpha_0}^{+}\right) \Psi\left(u_{\beta_0}^{-}\right) \Psi\left(K_{\beta_0'}\right).
\end{aligned} \tag{4.5}$$

Proof. In the left-hand side of (4.5) we shall deal with short exact sequences of the form

$$0 \to V_{\beta_0} \to V_{\lambda_0} \to V_{\alpha_0'} \to 0 \quad \text{and} \quad 0 \to V_{\alpha_0'} \to V_{\lambda_0'} \to V_{\alpha_0} \to 0$$

while in the right-hand side of (4.5) we shall deal with short exact sequences of the form

$$0 \to V_{\beta_0'} \to V_{\lambda_0} \to V_{\alpha_0} \to 0 \quad \text{and} \quad 0 \to V_{\beta_0} \to V_{\lambda_0'} \to V_{\beta_0'} \to 0.$$

Case (a). It is easy to see that

$$\text{r.h.s. of (4.5)} = a_{\lambda} a_{\lambda'} v^{\langle \lambda', \lambda' \rangle + \langle \lambda, \lambda' \rangle} \sum_{\substack{t\alpha \in \mathcal{B}(\Lambda, \mathcal{F}) \\ \beta \in \mathcal{B}(\Lambda, \mathcal{T})}} \mathbf{F}_{\lambda \lambda'}^{\alpha + \beta} u_{\alpha+\beta}^{+} K_{\lambda'}.$$

In the left-hand side of (4.5), we may assume that $\alpha_0 = T(\alpha)$ and $\beta_0 = T(\beta)$ with $\alpha \in \mathcal{B}(\Lambda, \mathcal{F})$, $\beta \in \mathcal{B}(\Lambda, \mathcal{T})$. Moreover, $\Psi(K_{\alpha_0'}) = K_{\lambda'-\alpha}$. So,

$$\text{l.h.s. of (4.5)} = \sum_{\alpha_0,\beta_0,\alpha_0'} v^D a_{\alpha_0} a_{\beta_0} a_{\alpha_0'} \mathbf{F}^{\lambda_0}_{\alpha_0'\beta_0} \mathbf{F}^{\lambda_0'}_{\alpha_0\alpha_0'} u^+_{\alpha+\beta} K_{\lambda'},$$

where

$$D = \langle \alpha_0, \alpha_0' \rangle - \langle \beta_0, \alpha_0' \rangle + \langle \alpha, \alpha \rangle + (\alpha, \beta) + \langle \alpha, \beta \rangle = \langle \lambda', \lambda' \rangle + \langle \lambda, \lambda' \rangle - 2\langle \alpha_0, \beta_0 \rangle.$$

So, by (3.4), in this case (4.5) holds. In a similar way, we can show (4.5) for the case (b).

Case (c). In the left-hand side of (4.5) we may assume $\beta_0 = T(\beta) \in \mathcal{B}(\Gamma, \mathcal{Y})$ and $\alpha_0' = T(\alpha') \in \mathcal{B}(\Gamma, \mathcal{Y})$ with $\beta, \alpha' \in \mathcal{B}(\Lambda, \mathcal{T})$. Moreover, we set

$$\alpha_0 = T(y) + T(x) \quad \text{with } y \in \mathcal{B}(\Lambda, \mathcal{T}) \text{ and } x \in \mathcal{B}(\Lambda, \mathcal{F}).$$

Thus, we have that

$$\begin{aligned} \text{l.h.s. of (4.5)} = & \sum_{y,x,\beta,\alpha'} v^{\langle y-x,\alpha' \rangle - \langle \beta,\alpha' \rangle + \langle y,x \rangle + \langle x,\beta \rangle + \langle x,x \rangle + (x,\beta)} \\ & \times a_{T(y)+T(x)} a_\beta a_{\alpha'} \mathbf{F}^{\lambda}_{\alpha'\beta} \mathbf{F}^{T(\lambda')}_{T(y)+T(x)T(\alpha')} u^-_y u^+_{x+\beta} K_{x-\alpha'}. \end{aligned}$$

Note that we have

$$\sum_{\alpha'} \sum_{\gamma} \mathbf{F}^{\lambda}_{\alpha'\beta} \mathbf{F}^{\alpha'}_{\gamma x} = \sum_{\gamma} \sum_{\alpha'} \mathbf{F}^{\lambda}_{\alpha'\beta} \mathbf{F}^{\alpha'}_{\gamma x} = \sum_{\gamma} \mathbf{F}^{\lambda}_{\gamma x \beta} = \sum_{\gamma} \mathbf{F}^{\lambda}_{\gamma x+\beta}$$

since the torsion theory $(\mathcal{T}(T), \mathcal{F}(T))$ is splitting. Moreover, we have that

$$a_{x+y} = a_x a_y \left| \operatorname{Hom}_\Lambda(V_x, V_\beta) \right| = a_x a_y v^{2\langle x,\beta \rangle}.$$

By (3.5), we have that (note that $K_{x-\alpha'} = K_\gamma$)

$$\text{l.h.s. of (4.5)} = \sum_{x,y,\beta,\gamma} v^D a_\gamma a_y a_{x+\beta} \mathbf{F}^{\lambda}_{\gamma x+\beta} \mathbf{F}^{\lambda'}_{y\gamma} u^-_y u^+_{x+\beta} K_{-\gamma},$$

where

$$\begin{aligned} D = & \langle y, \gamma + x \rangle - \langle x, \gamma + x \rangle - \langle \beta, \gamma + x \rangle + \langle y, x \rangle + \langle x, \beta \rangle + \langle x, x \rangle + (x, \beta) \\ & - 2\langle y, x \rangle - 2\langle x, \beta \rangle = \langle y, \gamma \rangle - \langle x + \beta, \gamma \rangle. \end{aligned}$$

Thus, we have that

$$\text{l.h.s. of (4.5)} = \sum_{\gamma_1,y,\gamma} v^{\langle y,\gamma \rangle - \langle \gamma_1,\gamma \rangle} a_\gamma a_y a_{\gamma_1} \mathbf{F}^{\lambda}_{\gamma\gamma_1} \mathbf{F}^{\lambda'}_{y\gamma} u^-_y u^+_{\gamma_1} K_{-\gamma}. \tag{4.6}$$

In the right-hand side of (4.5) we may assume $\beta_0 = T(\beta) \in \mathcal{B}(\Gamma, \mathcal{Y})$ and $\beta_0' = T(\beta') \in \mathcal{B}(\Gamma, \mathcal{Y})$ with $\beta, \beta' \in \mathcal{B}(\Lambda, \mathcal{T})$. Moreover, we set

$$\alpha_0 = T(y) + T(x) \quad \text{with } y \in \mathcal{B}(\Lambda, \mathcal{T}) \text{ and } x \in \mathcal{B}(\Lambda, \mathcal{F}).$$

Thus, we have that

$$\begin{aligned} \text{r.h.s. of (4.5)} \;=\; & \sum_{y,x,\beta,\beta'} v^{\langle y-x,\beta'\rangle - \langle \beta,\beta'\rangle + \langle y,x\rangle + \langle x,\beta\rangle + \langle x,x\rangle + (x,\beta)} \\ & \qquad \times a_{T(y)+T(x)} a_\beta a_{\beta'} \mathbf{F}^{\lambda'}_{\beta'\beta} \mathbf{F}^{T(\lambda)}_{T(y)+T(x)T(\beta')} u_y^+ u_{x+\beta}^- K_{-x+\beta'}. \end{aligned}$$

Note that we have

$$\sum_{\beta'} \sum_{\gamma'} \mathbf{F}^{\lambda'}_{\beta'\beta} \mathbf{F}^{\beta'}_{\gamma' x} = \sum_{\gamma'} \sum_{\beta'} \mathbf{F}^{\lambda'}_{\beta'\beta} \mathbf{F}^{\beta'}_{\gamma' x} = \sum_{\gamma'} \mathbf{F}^{\lambda'}_{\gamma' x\beta} = \sum_{\gamma'} \mathbf{F}^{\lambda'}_{\gamma' x+\beta}$$

since the torsion theory $(\mathcal{T}(T), \mathcal{F}(T))$ is splitting. Moreover, we have that

$$a_{x+y} = a_x a_y \left|\mathrm{Hom}_\Lambda(V_x, V_\beta)\right| = a_x a_y v^{2\langle x,\beta\rangle}.$$

By (3.4), we have that (note that $K_{-x+\beta'} = K_{\gamma'}$)

$$\text{r.h.s. of (4.5)} = \sum_{x,y,\beta,\gamma'} v^D a_{\gamma'} a_y a_{x+\beta} \mathbf{F}^{\lambda'}_{\gamma' x+\beta} \mathbf{F}^{\lambda}_{y\gamma'} u_y^+ u_{x+\beta}^- K_{\gamma'},$$

where

$$\begin{aligned} D \;=\; & \langle y, \gamma' + x\rangle - \langle x, \gamma' + x\rangle - \langle \beta, \gamma' + x\rangle + \langle y, x\rangle + \langle x, \beta\rangle + \langle x, x\rangle + (x, \beta) \\ & - 2\langle y, x\rangle - 2\langle x, \beta\rangle = \langle y, \gamma'\rangle - \langle x + \beta, \gamma'\rangle. \end{aligned}$$

Thus, we have that

$$\text{r.h.s. of (4.5)} = \sum_{\gamma_1, y, \gamma'} v^{\langle y,\gamma'\rangle - \langle \gamma_1,\gamma'\rangle} a_{\gamma'} a_y a_{\gamma_1} \mathbf{F}^{\lambda'}_{\gamma'\gamma_1} \mathbf{F}^{\lambda}_{y\gamma'} u_y^+ u_{\gamma_1}^- K_{\gamma'}. \tag{4.7}$$

By (2.4) and comparing (4.6) and (4.7), we see that in this case (4.5) holds. In a similar way, by using (3.5) we can show (4.5) for the case (d). □

Proof of Theorem 4.1. By (4.4) and Theorem 3.1, it is easy to see that Ψ respects relations (4.1)–(4.2). By Lemmas 4.1 and 3.2 in [10], Ψ respects the relation (4.3). So, Ψ defines an algebra morphism from $\mathcal{D}(\Gamma)$ to $\mathcal{D}(\Lambda)$. By an entirely similar argument, we can define an algebra morphism Ψ' from $\mathcal{D}(\Lambda)$ to $\mathcal{D}(\Gamma)$ such that $\Psi'\Psi = 1$. This completes the proof. □

5. The automorphism of $\mathcal{D}(\Lambda)$ associated to AR translation

For general Auslander–Reiten theory we refer to [3]. We have the functor $\mathrm{D} = \mathrm{Hom}_\Lambda(-,k)$; the *Nakayama* functor $\nu = \mathrm{D}\,\mathrm{Hom}_\Lambda(-,\Lambda)$ with inverse $\nu^- = \mathrm{Hom}(\mathrm{D}(-),\Lambda)$; the Auslander–Reiten translation $\tau = \mathrm{D}\,\mathrm{Ext}^1_\Lambda(-,\Lambda)$ with quasi-inverse $\tau^- = \mathrm{Ext}^1_\Lambda(\mathrm{D}-,\Lambda)$. Set $\mathcal{B}(\Lambda,\mathcal{I}) = \{\alpha \in \mathcal{B}(\Lambda) \mid V_\alpha$ is injective$\}$ and $\overline{\mathcal{B}}(\Lambda,\mathcal{I}) = \{\alpha \in \mathcal{B}(\Lambda) \mid V_\alpha$ has no injective direct summands$\}$. For $\alpha \in \overline{\mathcal{B}}(\Lambda,\mathcal{I})$ we denote by $\tau^-\alpha$ the isoclass of $\tau^- V_\alpha$, and for $\alpha \in \mathcal{B}(\Lambda,\mathcal{I})$ we denote by $\nu^-\alpha$ the isoclass of $\nu^- V_\alpha$. Then we have the following

Lemma 5.1. (a) *For $\lambda,\beta \in \mathcal{B}(\Lambda,\mathcal{I})$ and $\lambda',\alpha \in \overline{\mathcal{B}}(\Lambda,\mathcal{I})$, we have that*

$$\mathbf{F}^{\alpha+\beta}_{\lambda\lambda'} a_\lambda a_{\lambda'} = v^{-2\langle\alpha,\beta\rangle} a_\alpha a_\beta \sum_{\beta'} \mathbf{F}^{\nu^-\lambda}_{\beta'\nu^-\beta} \mathbf{F}^{\tau^-\lambda'}_{\tau^-\alpha\beta'} a_{\beta'}.$$

(b) *For $\lambda,\beta \in \overline{\mathcal{B}}(\Lambda,\mathcal{I})$ and $\lambda',\alpha \in \mathcal{B}(\Lambda,\mathcal{I})$, we have that*

$$\mathbf{F}^{\nu^-\alpha+\tau^-\beta}_{\tau^-\lambda\nu^-\lambda'} a_\lambda a_{\lambda'} = v^{-2\langle\alpha,\beta\rangle} a_\alpha a_\beta \sum_{\alpha'} \mathbf{F}^{\lambda}_{\alpha'\beta} \mathbf{F}^{\lambda'}_{\alpha\alpha'} a_{\alpha'}.$$

(c) *For $\lambda,\alpha',\beta_2 \in \mathcal{B}(\Lambda,\mathcal{I})$ and $\beta_1 \in \overline{\mathcal{B}}(\Lambda,\mathcal{I})$, we have that*

$$\mathbf{F}^{\lambda}_{\alpha'\beta_1+\beta_2} a_{\alpha'} a_{\beta_1+\beta_2} = v^{2\langle\beta_1,\beta_2\rangle} a_{\beta_1} a_{\beta_2} \sum_{\gamma} \mathbf{F}^{\nu^-\lambda}_{\nu^-\gamma\nu^-\beta_2} \mathbf{F}^{\nu^-\alpha'}_{\tau^-\beta_1\nu^-\gamma} a_\gamma.$$

(d) *For $\lambda',\alpha',\alpha_1 \in \overline{\mathcal{B}}(\Lambda,\mathcal{I})$ and $\alpha_2 \in \mathcal{B}(\Lambda,\mathcal{I})$, we have that*

$$\mathbf{F}^{\lambda'}_{\alpha_1+\alpha_2\alpha'} a_{\alpha'} a_{\alpha_1+\alpha_2} = v^{2\langle\alpha_1,\alpha_2\rangle} a_{\alpha_1} a_{\alpha_2} \sum_{\gamma} \mathbf{F}^{\tau^-\lambda'}_{\tau^-\alpha_1\tau^-\gamma} \mathbf{F}^{\tau^-\alpha'}_{\tau^-\gamma\nu^-\alpha_2} a_\gamma.$$

Proof. (a) We have an equivalence between the subcategory

$$\{V_\alpha \mid \alpha \in \mathcal{B}(\Lambda,\mathcal{I})\} \cup \{\mathbf{T}V_\beta \mid \beta \in \overline{\mathcal{B}}(\Lambda,\mathcal{I})\}$$

of $D^b(\Lambda)$ and $\mathrm{mod}\,\Lambda$ given by $V_\alpha \mapsto V_{\nu^-\alpha}$ and $\mathbf{T}V_\beta \mapsto V_{\tau^-\beta}$ for $\alpha \in \mathcal{B}(\Lambda,\mathcal{I})$ and $\beta \in \overline{\mathcal{B}}(\Lambda,\mathcal{I})$. Thus, (a) follows by Theorem 1.1. (b) is the dual version of (a). (c) and (d) can be shown directly. □

Now we consider the map $\tau^-:\mathcal{D}(\Lambda) \to \mathcal{D}(\Lambda)$ as follows. For $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in \overline{\mathcal{B}}(\Lambda,\mathcal{I}), \alpha_2 \in \mathcal{B}(\Lambda,\mathcal{I})$, set

$$u^\pm_\alpha \mapsto v^{-\langle\alpha,\alpha_2\rangle} u^\pm_{\tau^-\alpha_1} K_{\mp\nu^-\alpha_2} u^\mp_{\nu^\mp\alpha_2}, \qquad K_\gamma \mapsto K_{\mathrm{g}(\gamma)} \quad \text{for } \gamma \in K_0(\Lambda), \tag{5.1}$$

where the map $\mathrm{g}: K_0(\Lambda) \to K_0(\Lambda)$ is given by

$$\mathrm{g}\big(\dim V_\gamma\big) = \dim V_{\tau^-\alpha} - \dim V_{\nu^-\beta}.$$

By an argument completely similar to Theorem 4.1, we have the following

Theorem 5.1 (Sevenhant–Van den Bergh, cf. [10]). *There is an automorphism τ^- of $\mathcal{D}(\Lambda)$ given by* (5.1).

By a direct computation, we have the following

Proposition 5.1. *As automorphisms of $\mathcal{D}(\Lambda)$, $\tau^-\omega = \omega\tau^-$, where ω is given by* (2.5).

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